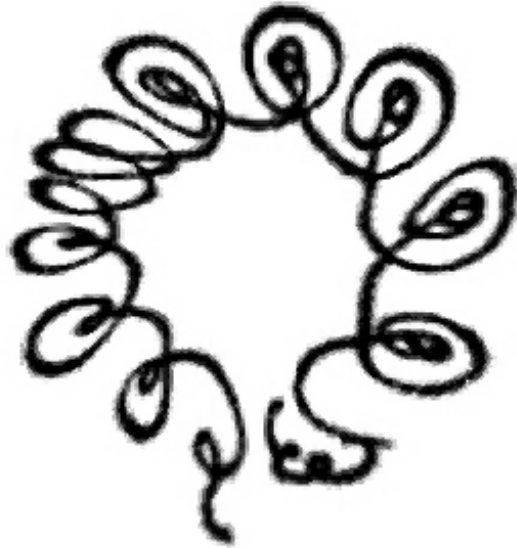


Baroque temperaments



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20 November 2016

Introduction

Baroque keyboard temperaments are explained. I will show how to visualize a temperament theoretically using the circle of fifths and how this will allow you to determine at a glance how the major thirds and other intervals sound in various keys in a particular temperament. Several common and uncommon temperaments are discussed and evaluated. How to tune these by ear and/or computer is also discussed. Finally I show how you can design your own temperament aurally at the keyboard, or theoretically.

The approach presented here differs from the usual methods in the literature which usually present extensive tables and spreadsheets with hundreds of numbers which are not useless for theoretical purposes, but the goal here is to show analysis tools that serve the purpose of the practicing musician which is to play music that sounds good.

The instrument to which all this applies can be organ, harpsichord, clavichord, MIDI, piano, and fretted string instruments.

Temperaments

If we start tuning a keyboard from, say, middle C by tuning pure fifths (P5s) or their inversions, pure fourths (P4s), we find that when we tune the last P5, F-C, the resulting C is about 24 cent (23.46) higher than where we started off. This is about $\frac{1}{4}$ of a semitone, and is called a (Pythagorean) "comma". So in order to close the circle of fifths we have to narrow some or all of the P5s in such a manner that the sum of all the narrowings (optional widenings counted negative) over all twelve P5s adds up to a comma, or 24 cents. Any specific way of doing this is called a temperament, and there are infinitely many.

Another consideration is the M3s (major thirds). If we tune 3 pure M3s from C, the resulting last C is 41 cents flat (a diesis). So clearly some or all M3s have to be widened.

Furthermore, a M3 spanned by pure fifths (e.g. F-CGD -A) is very much out of tune; this is called a Pythagorean M3. It is generally considered to be at the limit of what is acceptable. Such a M3 is wide, and if widened further it will be so bad that it cannot be used at all. It is wide by a syntonic comma, which is 21.5 cent. By coincidence, this is quite close to the Pythagorean comma, and for practical tuning and evaluation of temperaments the 2 cent difference between the two can often be neglected, which makes analysis a lot easier. For all practical purposes, though not exactly, a syntonic comma is $\frac{11}{12}$ of a Pythagorean comma.

As we have to narrow the P5s anyway, how much do we have to narrow 4 consecutive P5s (e.g., F CGD A) to get a pure M3? The answer is $\frac{1}{4}$ syntonic comma. This is remarkable, because it means that if we tune 4 consecutive P5s $\frac{1}{4}$ ' (the apostrophe stand for a comma) narrow it results in a pure M3 and also closes the circle of fifths, if we neglect the difference between the syntonic and Pythagorean commas. Mathematically speaking this is an error, but the magnitude of the error is close to the errors we make anyway in practical tuning.

Usually the minor thirds (m3s) are ignored in analysing temperaments and we shall not consider them.

Pythagorean, equal and Kirnberger 3 temperaments and the circle of fifths

Here I will explain how to use circle of fifths diagrams to evaluate temperaments using three examples of simple temperaments.

If we don't care about M3s the obvious thing to do would be to tune all but one P5 pure. This is the Pythagorean temperament, popular in the Medieval period. If we don't care about anything we could just make all P5s equally narrow ($1/12'$ or 2 cents) resulting in equal temperament (ET). A third obvious choice is to select one M3 (say CE) to be pure, and narrow the four P5s that span it by $1/4'$ (a syntonic comma to be precise), resulting in the Kirnberger 3 (KB3) temperament. These are the most straightforward temperaments.

The best way to appreciate them is to tune your instrument and listen, but it is also possible to evaluate them theoretically by drawing a "circle of fifths" diagram. Some theoretical skills are required when designing temperaments, as the number of possible temperaments is infinite, so it is impractical to try them all. I use the free program Scala to generate the circle of fifths diagrams.

To read these diagrams, all that is relevant are the P5s and M3s, so ignore the minor 3ds (blue lines) and the harmonic 7th (orange rectangles). The size of the P5s are indicated in cents. In Fig. 1 (Pythagorean tuning) we see one P5 narrow by 24 cents (rounded) which is one comma, for ET (Fig. 2) each P5 is $1/12'$ narrow, which is about 2 cents. For KB3 (Fig. 3) CE is a pure M3, and this can practically be achieved by making the P5s CGDAE all $1/4'$ (6 cent) narrow. Because of the difference between the syntonic and Pythagorean comma, the P5s actually have to be a little less narrow (5.4 cents) and this results in one more P5 needing to be narrowed to close the circle. In practice we tune CE pure and then distribute the "error" over the rest of the P5s in an imperceptible manner. In Fig. 3 the "error" is all put into the single P5 G#-Eb, but in practical tunings it would be distributed over the 8 P5s from E to C, theoretically all 0.25 cent narrow, which no one can hear. Note that in Fig.1 we see several almost pure M3s in remote keys. They are not exactly pure because of the difference between the syntonic and Pythagorean commas. The pure M3s are in remote keys and are not used anyway in medieval music for which Pythagorean temperament is used.

In these diagrams the M3 quality is plotted in red. If it is in the centre as for CE in Fig. 3 it is pure. If it is on the outer boundary as for example CE in Fig.1 it is Pythagorean, i.e., at the limit of sounding acceptable (though this is somewhat subjective). The green dotted circle is a landmark indicating $1/6'$ narrow P5s. A M3 on this line is slightly wider than in ET. Fig. 2 indicates the quality of the M3s in ET.

If we didn't have the red lines we could figure it out for ourselves. In Fig. 1 the total narrowing of the P5s spanning for example CE is 0, so this is a Pythagorean M3. The total narrowing of the P5s spanning CE in ET is 6 cent, or $1/3'$. In Fig. 3 the CE P5s add up to 21.6 cent, or about $11/12'$ giving a pure M3.

To continue the exercise, let's try to figure out how the M3s on A and D are in KB3 compared to ET. The narrowness of the P5s spanning DF# is 10.8 cent (ET 8 cent), for AC# 5.4 cent (ET also 8 cent of course). So DF# is better than in ET, and AC# is worse than in ET.

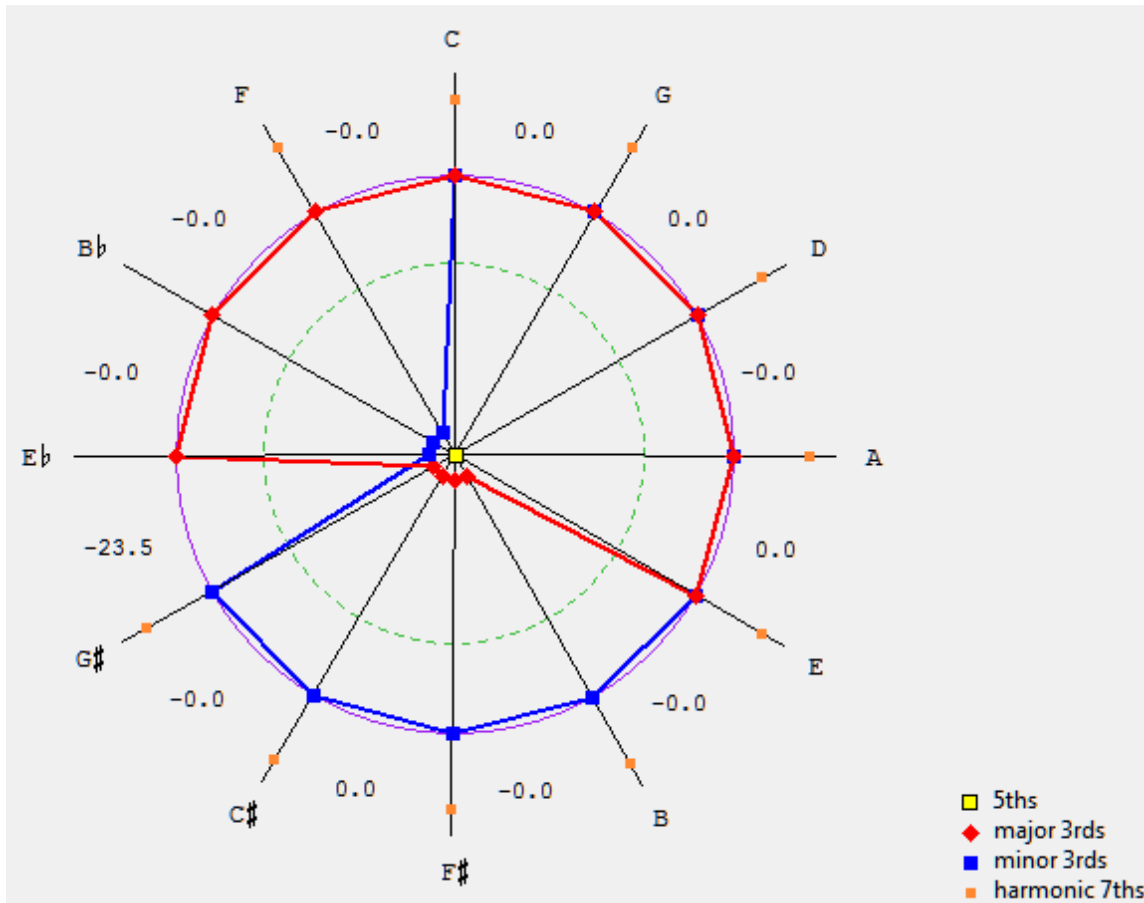


Figure 1. Pythagorean tuning. All P5s are pure except G#Eb, which is so bad it is called the “wolf”.

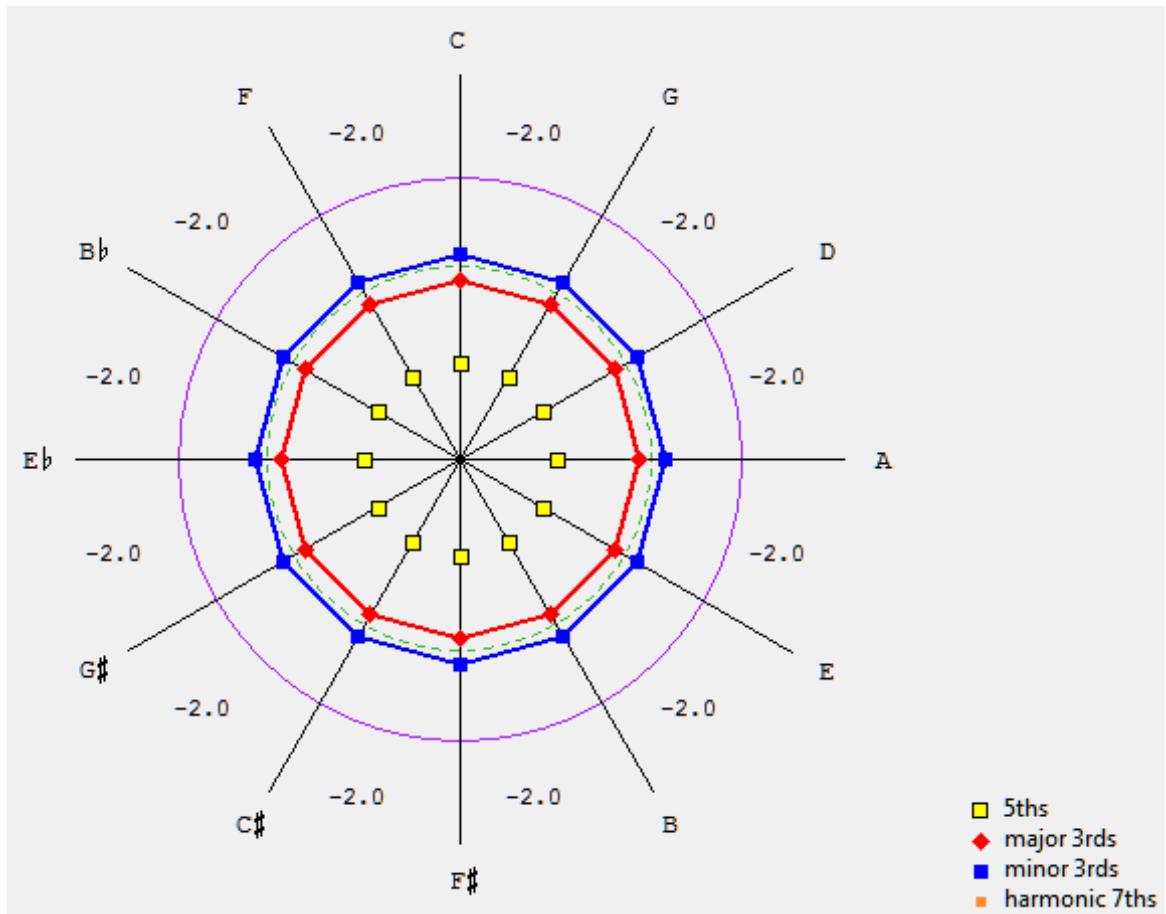


Figure 2. Equal temperament tuning. All P5s the same.

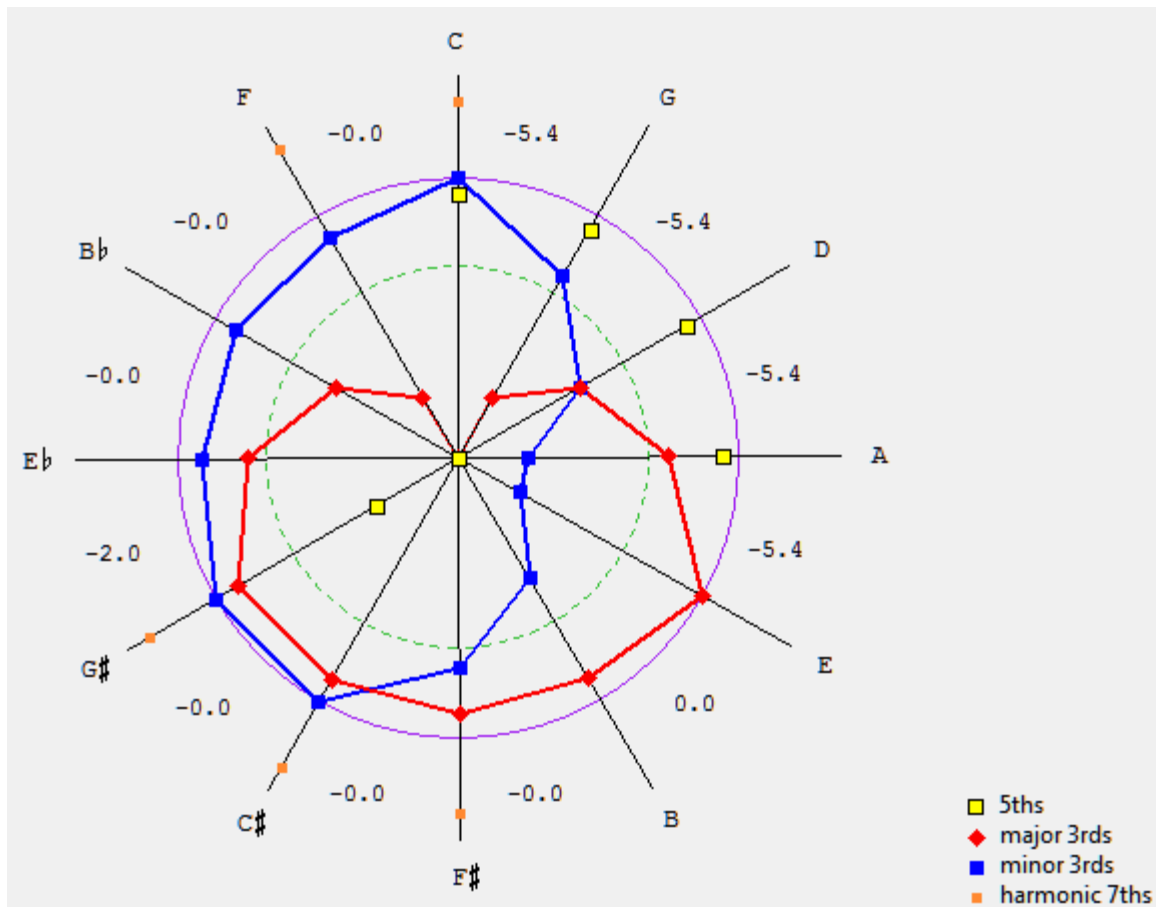


Figure 3. Kirnberger 3.

Meantone temperament

Historically a fourth straightforward temperament preceded KB3 and ET, namely meantone (MT). The idea here is to make all P5s $\frac{1}{4}$ ' (syntonic, to be precise) narrow except one which has to be very wide to close the circle of P5s. It is depicted in Fig. 4. We see the M3s are all either pure or "off the scale", meaning completely useless. The P5s are all (but one) about $\frac{1}{4}$ ' narrow, which is about the limit of them still being perceived as "in-tune". The "wolf" P5 is G#Eb and is grossly out of tune. Note that as long as the bad intervals are avoided, music in MT will sound as key-neutral as ET, as all intervals are of the same quality. I am aware of only one major keyboard piece by Bach that is playable in this temperament, the prelude and fugue in F major from the Well Tempered Clavier Book 1. The wolf G#Eb does appear but it is really not a P5 but diminished 6th and sounds appropriate.

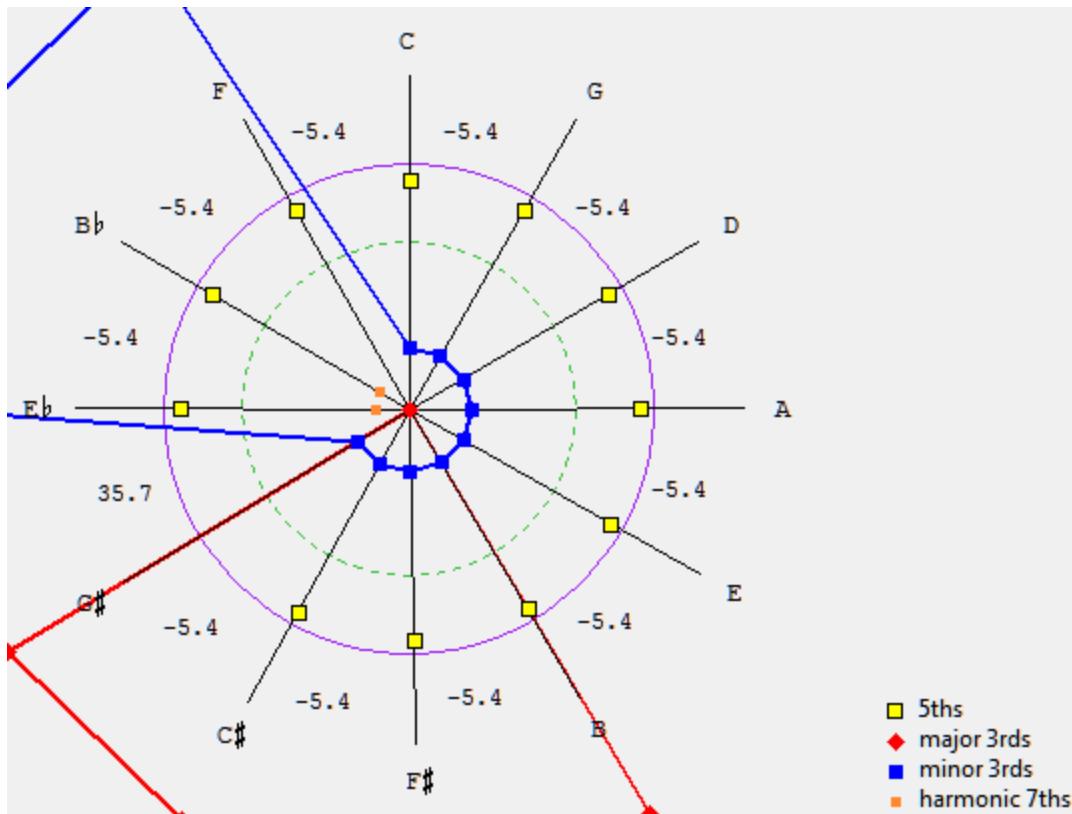


Figure 4. $\frac{1}{4}'$ meantone.

Well temperaments

A well temperament (WT) is a temperament where all P5s and M3s are usable, meaning no P5 can be off by more than $\frac{1}{4}'$ and no M3 can be worse than Pythagorean. P5s that are narrower than $\frac{1}{4}'$ sound objectionable, and the same for Pythagorean M3s. (This rule of thumb may be adjusted to personal preferences of course, but I will pretend these are hard rules for simplicity.)

Obviously KB3 qualifies, but it has some disadvantages in the pattern of M3s. Generally speaking for Baroque music the tonal quality is determined by the quality of the M3s, with the P5s playing no role as they are all pretty good. Examining Fig. 3 we note that CE is pure, FA, GB are equal and very good, BbD and DF# are equal and marginally better than ET, EbG and AC# are slightly worse than ET, and M3s on the rest of them are Pythagorean. One problem with this is that once you hit 4 accidentals, everything will start sounding the same. Another problem is that FA and CE are not the same, despite F major and C major both having no accidentals. (Bb is not considered an accidental historically, it is called H in German.) Yet another consideration is efficiency: the very good M3s on F and G are really good enough, the CE despite being beatless does not really sound any better in practical music.

These considerations lead us to Werckmeister 3 (WM3) (Fig. 5) which in some sense is the best of all well-temperaments of $\frac{1}{4}'$ strength. Ton Koopman tunes exclusively in this temperament for all Bach works, keyboard or otherwise. I here introduced the concept of "strength" of a WT. If we consider a WT a scheme whereby some P5s are detuned from pure by some amount, I call the maximum amount the

“strength” of the temperament. The weakest WT is ET, with a strength of $1/12'$. The strongest temperaments are $1/4'$. Stronger temperaments like $1/3'$ have been considered but a $1/3'$ narrow P5 is indigestible by the sensitive ear.

As we see in Fig. 5 instead of 4 consecutive P5s being all narrowed equally there are only 3, and the 4th one is placed on BF# apparently randomly. Actually, it is placed very cleverly. Let us start with the 3 $1/4'$ narrow P5s CGDA and figure out why the placement of the remaining $1/4'$ on BF# is so clever.

First notice that FA and CE are equal and both $3/4'$ narrowed w.r.t. Pythagorean, much better than $1/3'$ in ET. BbD and GB have only 2 narrow P5s but will sound a bit better than ET. EbG has only one $1/4'$ narrow P5 and will sound a bit worse than in ET (in practice about the same). Now if we did not have the narrow P5 on BF# the same story would hold for DF#, but D major should be better than Eb major. Why? Well, if only to please the trumpets. Fact is D major is much more common than Eb major. WM3 does the trick because the P5 on BF# make DF# the same as GB. Going further, if we didn't have the narrow P5 on BF#, AC# would be Pythagorean. As it is, it is the same as EbG. Even E and B have non-Pythagorean M3s. However AbC is Pythagorean. Why this preference for the sharps? Simply because modulation to the dominant is much more common than to the subdominant, obviously so in fugues. So it makes sense to favor sharps. Note that in WM3 you can play in E major and the dominant (B major) will still be better than Pythagorean.

Also note that in the “normal” Bach works (i.e., excluding the Well Tempered clavier and the transposition to B minor of the C minor French Overture, which has one piece in B major) the maximum number of accidentals is in major 4 sharps or 3 flats, and in minor 2 sharps or 4 flats.

Another issue is that M3s on A, E, and B sound the same, whereas it would probably be better if they gradually became wider. Finally, a $1/4'$ wide P4 is already at the limit of what is acceptable.

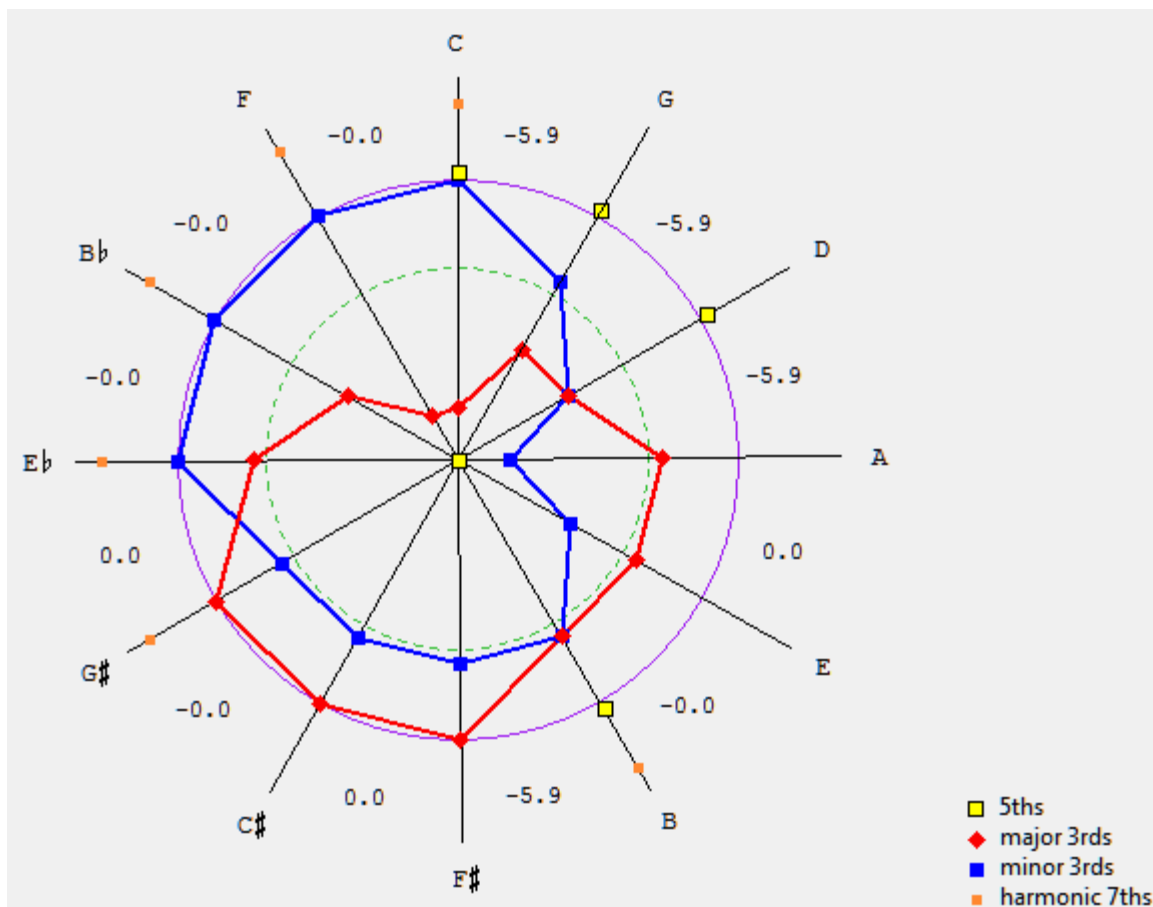


Figure 5. Werckmeister 3.

Improving WM3

Here I will discuss two modern attempts to slightly modify WM3 to improve it in some sense. These are the $1/5'$ Kellner and the $1/6'$ Barnes temperament. Before proceeding, we may wonder if we can't find a better $1/4'$ temperament; after all there are 495 possible ways to distribute 4 $1/4'$ narrow P5s over the circle of fifths. An example I made up is depicted in Fig. 6. There are no more Pythagorean M3s, but the temperament is not very differentiated. All keys with one accidental (not counting Bb) have better than ET M3s, and all the rest are imperceptibly worse than ET. It is left as an exercise for the reader to come up with other examples and figure out why such temperaments have never been proposed before as far as I know.

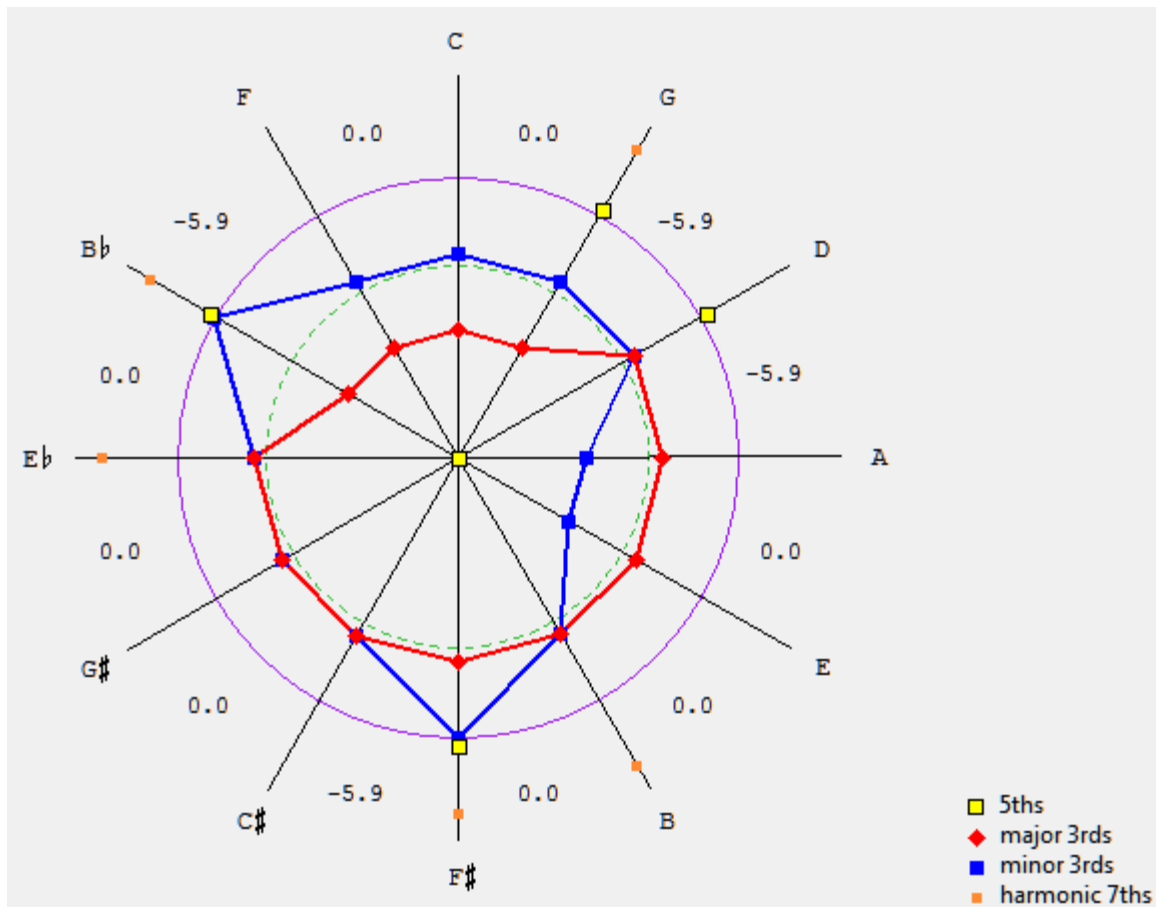
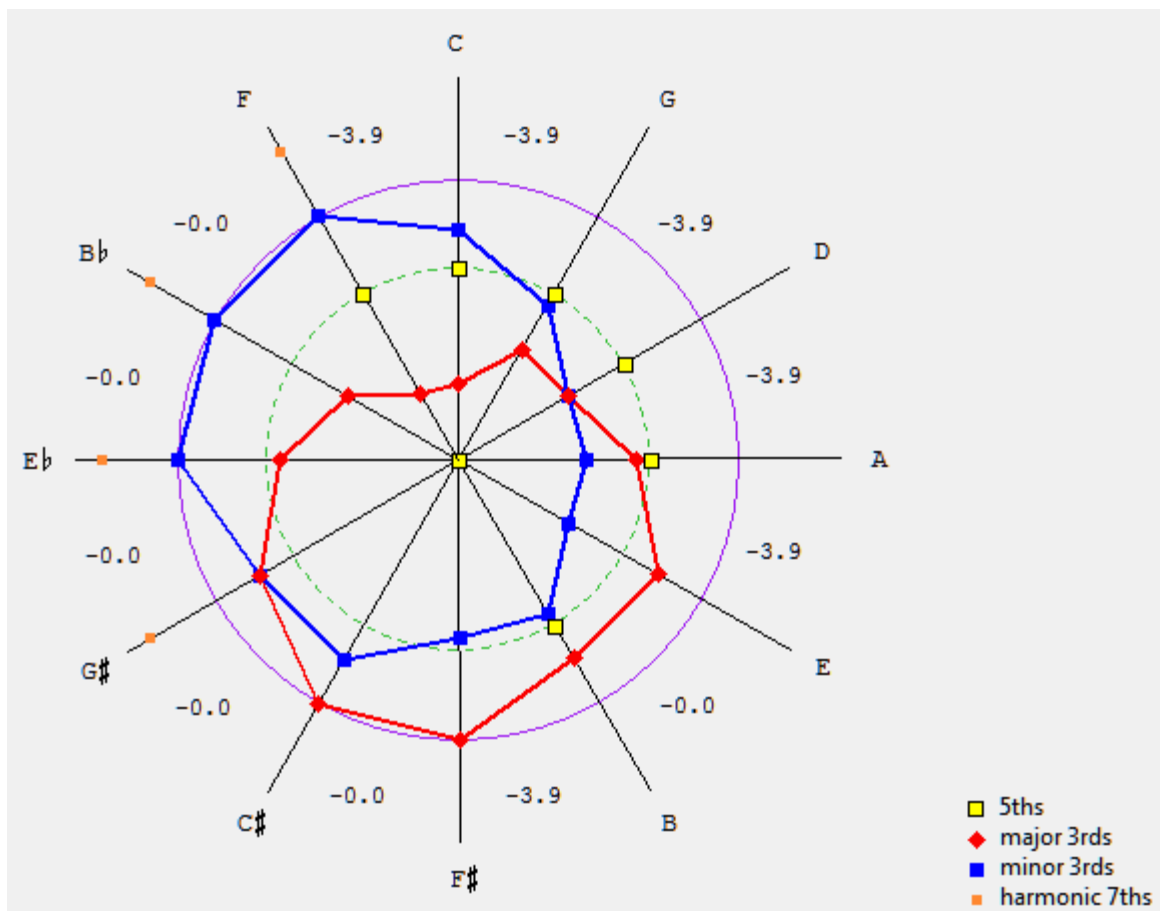


Figure 6. $\frac{1}{4}'$ temperament without Pythagorean M3s.

Next let us look at Barnes temperament, disregarding the flawed logic he used justifying it from counting intervals in the Well Tempered Clavier. (The author has counted all intervals in the WTC using software and found all M3s are used indiscriminately.) It is a $\frac{1}{6}'$ temperament and has narrow P5s in the same place as WM3 but added two more (as we have now six tempered P5s). As should be clear from Fig. 7, the increase in M3 widths is now somewhat smoother, the Ab M3 is no longer Pythagorean, and A and E are now differentiated. Of course the improved quality of these intervals has to come at the expense of some other intervals, and the best M3s FA and CE are now slight wider than in WM3. This is an excellent temperament which is surprisingly unpopular.



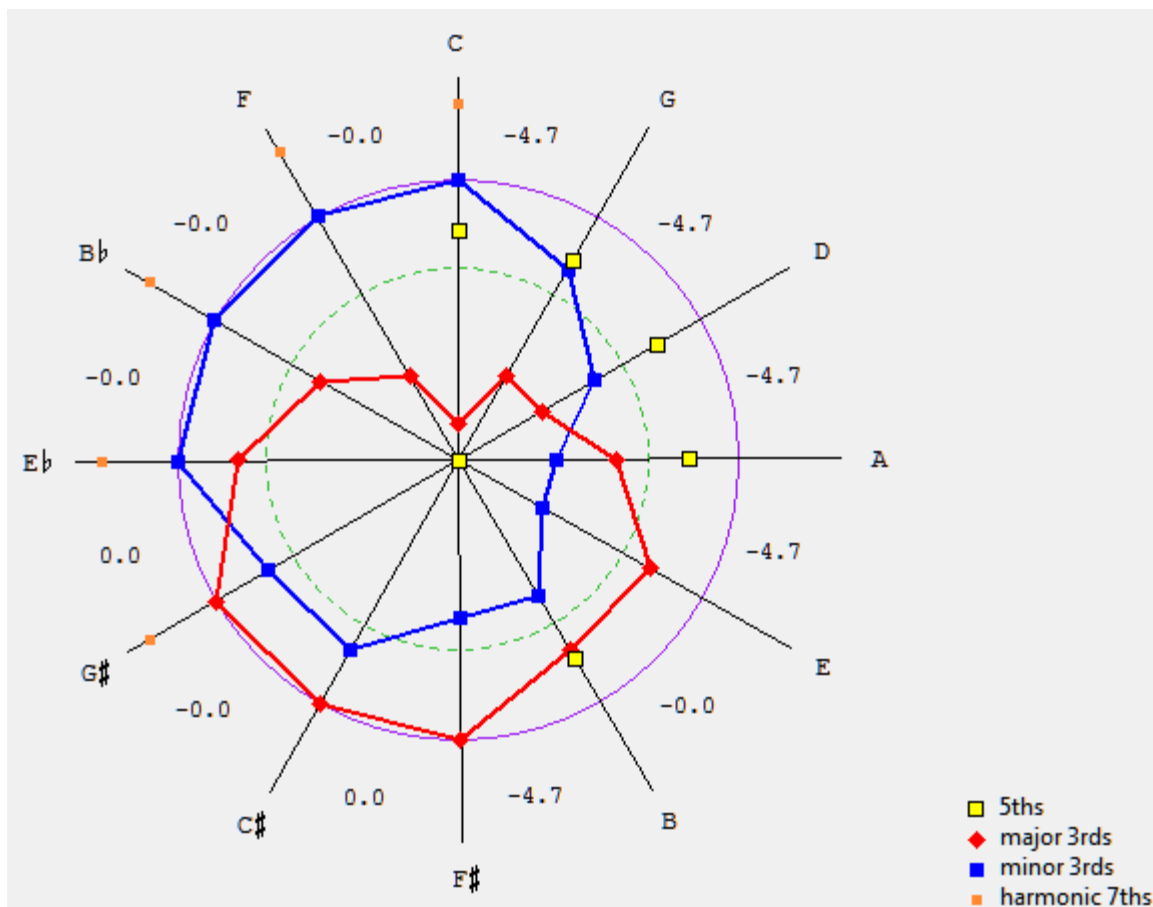


Figure 8. Kellner temperament.

Next let us look at Kellner's temperament, disregarding the flawed logic he used justifying it using numerology and German pomposity. It is identical to WM3 in the placement of 4 narrow P5s, and the fifth P5 is placed on AE. Its features are that now the CE is even better than in WM3, which is a bit pointless as it was already very good, but FA and CE are now different, which is not in accordance with the idea that Bb is not a flat, Bb and Eb have worse M3s than in WM3 or Barnes, AC# is slightly better. However like WM3 we still have three Pythagorean M3s. So unless you really care a lot about AC# the Kellner temperament is inferior to WM3 (and Barnes).

Other historical and modern temperaments

Many $1/6'$ temperaments have been proposed, some having both $1/6'$ and $1/12'$ narrow P5s. Let us start with the worst, which ironically also seems to be the most popular, the Vallotti/Young temperament, see Fig. 9. It is nicely symmetric, but there are 3 Pythagorean M3s, and one on B, which makes playing E major fugues somewhat unpleasant when the second voice comes in.

A better version is Young 2 (Fig. 10) which is the tuning of the Bach organ in Leipzig with one tiny modification, though it is advertised for nationalistic reasons as a modification of Neidhardt's temperament but this would require more than one modification. Young 2 has a nice progression of

M3s from good all the way up to one Pythagorean and its only flaws are that FA and CE are not equivalent and we would probably prefer a Pythagorean M3 on C# rather than on F#.

In Fig. 11 the tuning of the Bach organ in Leipzig is depicted. It differs from Young 2 in that the 1/12' narrow P5 on BbF in Young 2 has been moved to F#C#, resulting in the Pythagorean M3 now residing on C# as it should. The author considers this the second best temperament known, its only flaw being that FA and CE are not the same. However they are quite close and anyone claiming to be able to hear the difference should expect some scepticism.

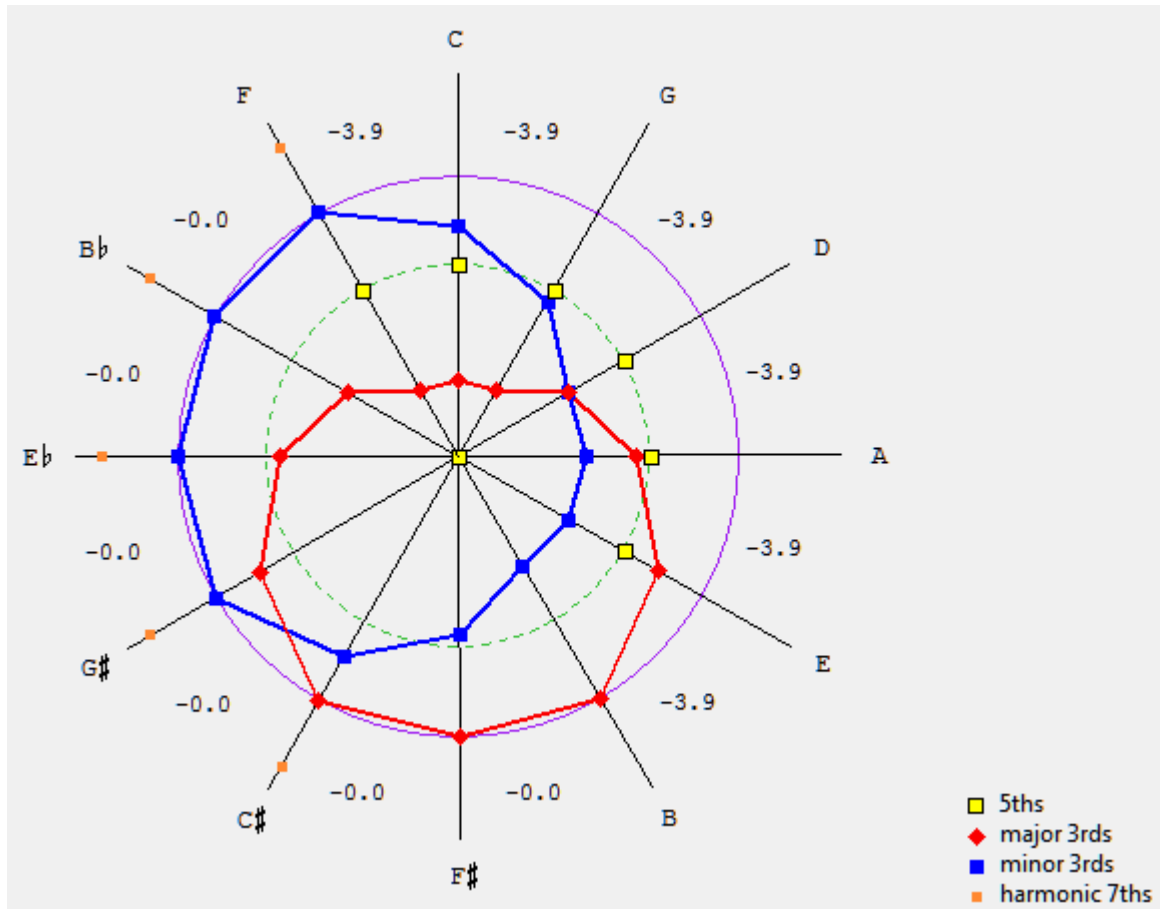


Figure 9. Vallotti temperament.

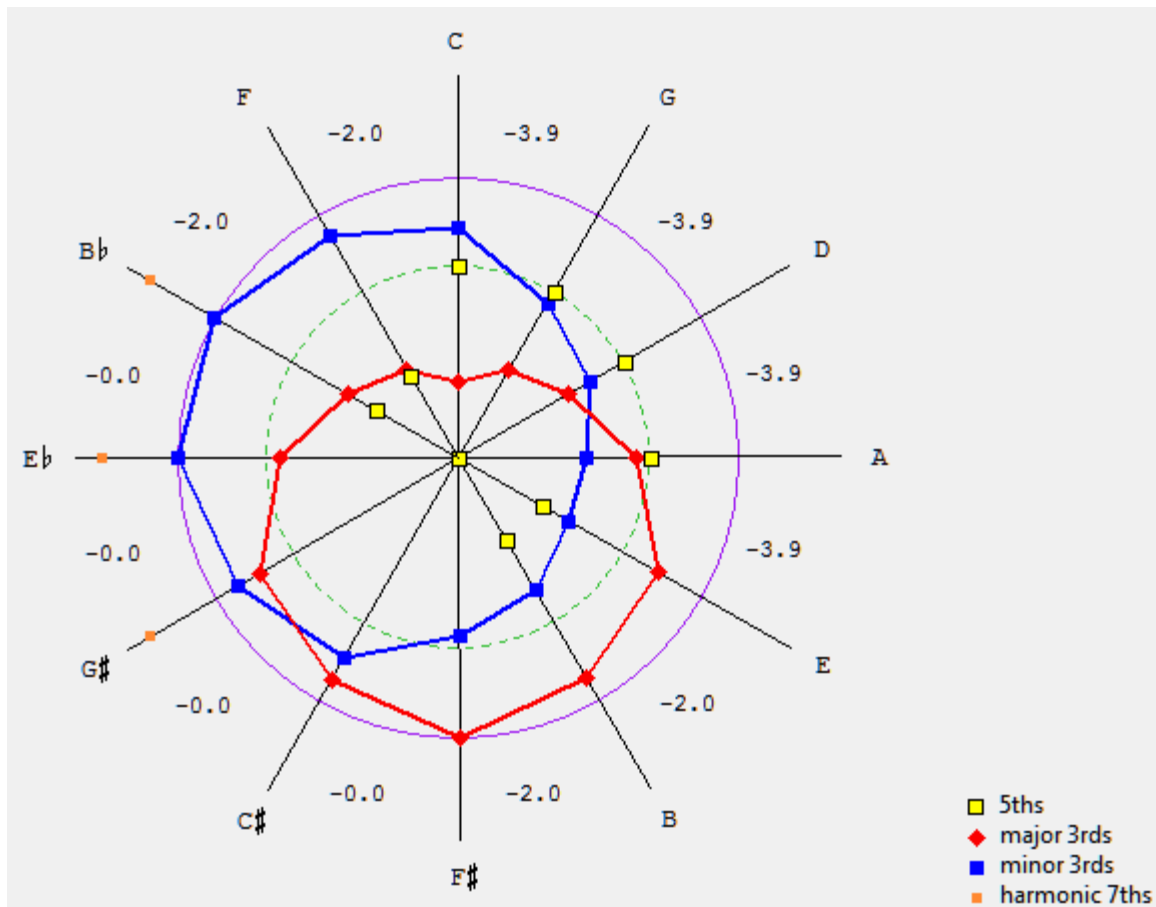


Figure 10. Young 2 temperament.

In Fig. 12 we show the Lehman-Bach temperament, designed from a particular interpretation of the ornaments on the top of the title page of the Well Tempered Clavier 1. I don't think this makes much sense as the worst key is E, and F# major will sound better than A major. However if you follow the spirit rather than the letter of Lehman's reasoning we arrive at what I call reverse Lehman-Bach 14, a $1/7'$ temperament, which is practically perfect in every way, see Fig. 13. I leave it as an exercise for the reader to appreciate how good this temperament is, as this should now be obvious.

Nevertheless, if you want something stronger with Pythagorean M3s, I like the temperament depicted in Fig. 14, which is as close to Young 2 as to Barnes, however the shape of the M3 curve is most like reverse Lehman-Bach 14, except with somewhat greater contrast. I call it Young2Kees.

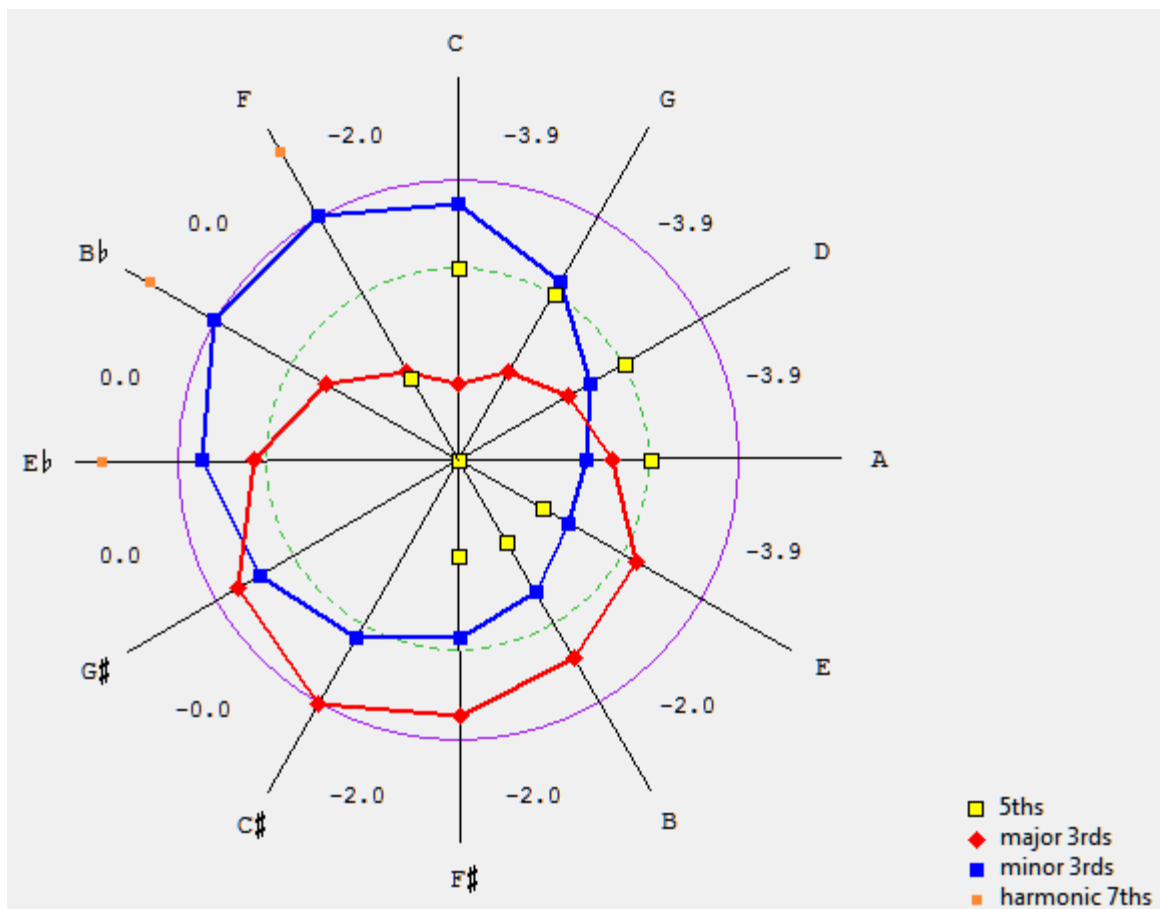


Figure 11. Temperament of the Bach organ in Leipzig.

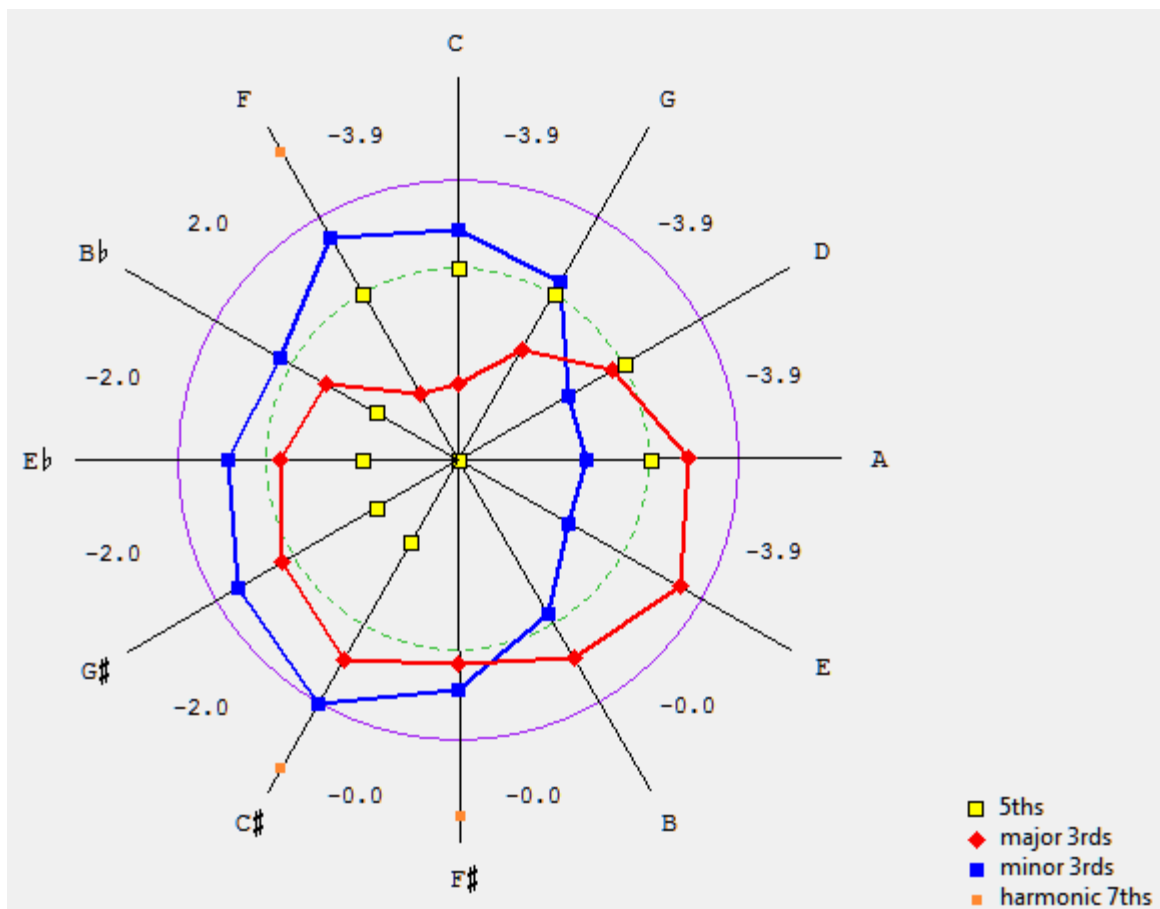


Figure 12. Lehman-Bach temperament.

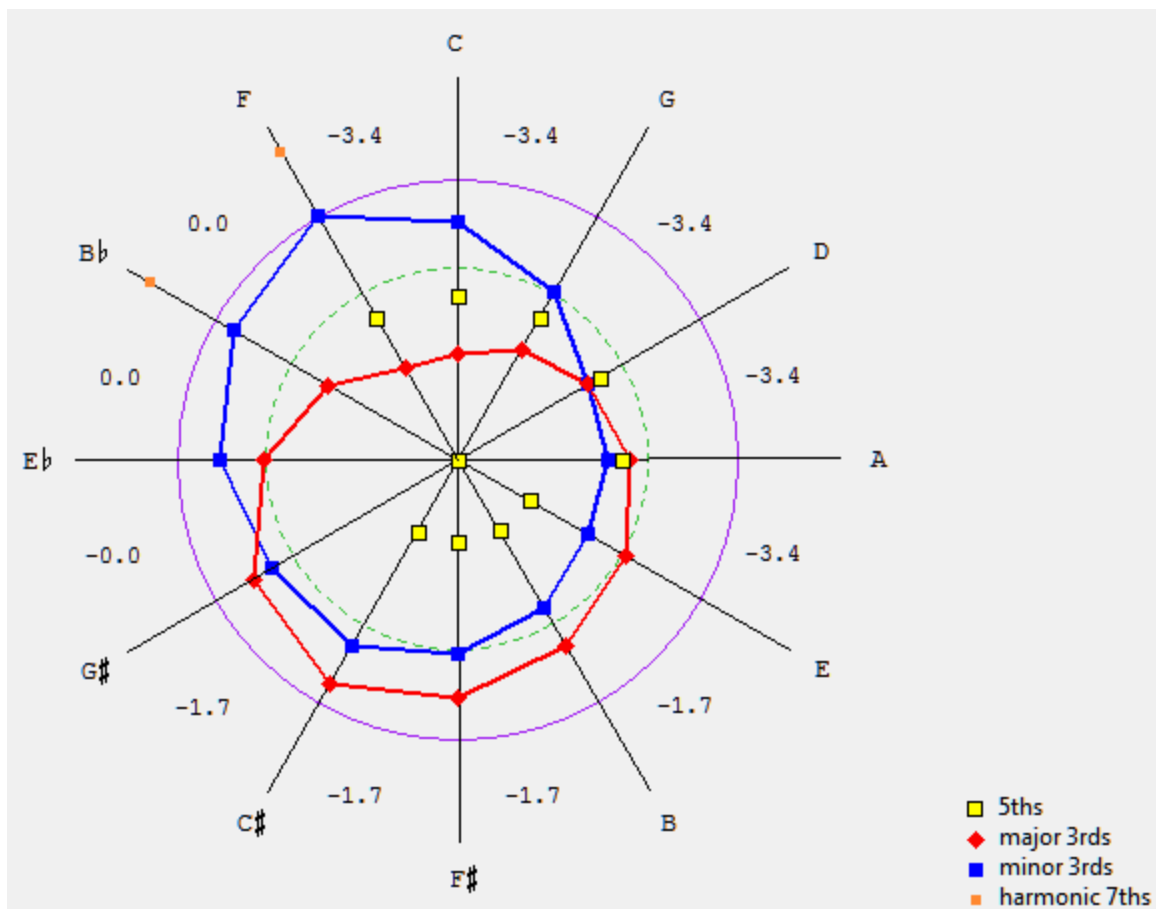


Figure 13. Reverse Lehman-Bach 14 temperament.

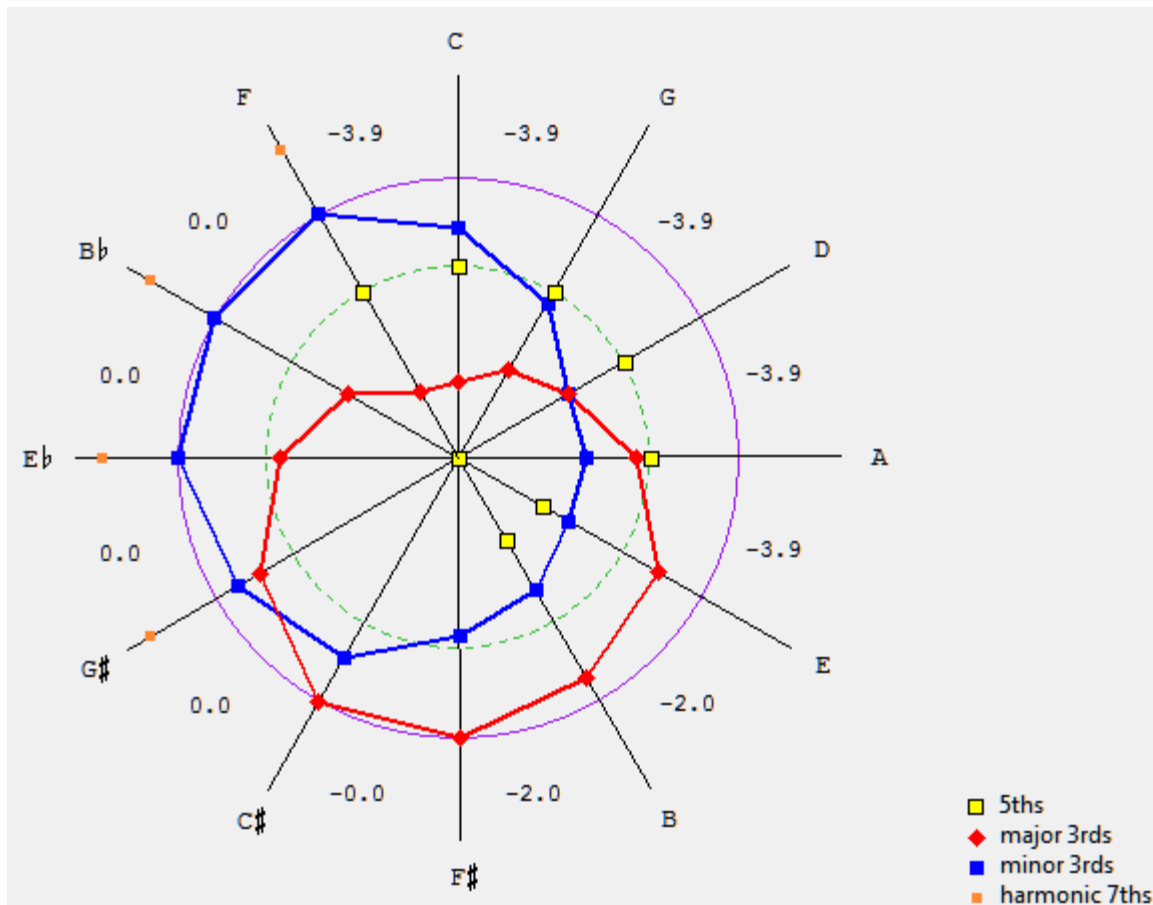


Figure 14. Young2Kees temperament.

Tuning temperaments electronically

The easiest way to tune temperaments is to use some electronic tuning device, like Tunelab. In this case you need to know the offsets in cent from ET for each of the 12 notes, assuming A fixed at your pitch standard. If you don't have these offsets you can figure them out by comparing to the ET diagram.

For example consider reverse Lehman-Bach 14 (Fig. 13) and compute the ET offset for C#. Going around the circle starting at A we add up the (negative) size of the P5s and get $-3.4-1.7-1.7-1.7 = -8.5$. Doing the same in ET (Fig. 2) we get $-2-2-2-2 = -8$. So the offset of C# is -0.5 cent. Repeating this for the other 10 notes gives you all the offsets.

Tuning temperaments by ear

To tune the temperaments by ear is less clear-cut and usually requires you to be able to detect beat rates (beats per second, bps) of intervals. The method I use is to set just the bps of F3A3 (F3 is the F defined by the bass clef) absolutely (e.g., 4 bps) and proceed by making some other specific beat rates (usually P5 and P4) about the same. Of course perfect P4 or P5s are easy. I prefer to tune the 12 notes in the range F3-E4, which is standard practice in piano tuning. In this range all beat speeds are moderate and easily heard. So all pitches below refer to this octave range unless otherwise indicated.

There is no single formula so I'll go through some examples.

Let's start with a simple example. First of all Pythagorean: just tune beatless clockwise from A to G# (i.e., A3E4B3F#3C#4G#3, and then beatless counter-clockwise from A to Eb (i.e., A3D4G3C4F3Bb3Eb4). Done.

Next take KB3. This is simple as you don't need to set any particular beat rate and you start from C. First tune E4 as a pure M3 from C4. Now tweak G, D, and A so that CG DG DA and DE are all about equally out of tune (the P5s narrow, the P4's wide). Strictly speaking they should all beat in specific ratios, but if you get them all to beat about equally fast that's good enough. Finally tune pure E-BF#C#G# and C-FBbEb.

For WM3 you can start as with KB3 tuning E temporarily as a pure CE and then fitting in G, D, and A (this is not theoretically exact due to the difference between the syntonic and Pythagorean commas, but good enough). After that, retune E pure to A. Then tune B pure to E and tune from C down (counter clockwise) the circle of fifths pure until F#. As a check, play F#B; it should sound the same as GC.

Reverse Lehman-Bach 14 is more complicated; you now have to start from A and tune F as a wide M3 from it with a specific beat speed of 4bps (MM=240). Then tune CGD so the P5s and P4's spanning FA all sound the same. Next tune E to A such that the P5 AE is the same as the P5 GD. After this go counter clockwise around the circle from F with pure P5 and P4s until G# (so FBbEbG# all pure). Finally adjust C#, F#, and B so that the P5s and P4s spanning EG# all sound the same.

Calculating absolute beat speeds

How do we know that FA should beat at 4 bps in Reverse Lehman-Bach 14? Recall that FA would be pure if the P5s spanning it had a total narrowing of a syntonic comma, which is 21.5 cent or 11/12' (Pythagorean comma from here on). In this case the numbers add up to 13.6 cent or (since those P5s are all narrow by 1/14') 2/7'. So the FA is 21.5-13.6 = 8 cent wide. Beating is caused by the misalignment of the 5th partial of F3 (A5~880Hz) and the 4th partial of A3 (also A5 of course). They are off by 8 cent. The formula for the beat speed is

$$\text{bps} = f \cdot c / 1731 \quad (\text{eq. 1})$$

with f the frequency of the partial (here 880 Hz) and c the discrepancy in cents, and for this example we get bps = 4/second.

Tweaking a temperament

Suppose you like Reverse Lehman-Bach 14, but would like FA to be a bit more pure and you decide to tune it at 3 bps instead of at 4 bps and then follow the tuning recipe? That's perfectly fine and results in another fine temperament that you can just tune and play. However let's analyse it. We can use eq. 1, plug in bps (=3) and calculate cent offset of FA as $c = 1731 \cdot \text{bps} / f$ which gives $1731 \cdot 3 / 880 = 5.9$. This is the cents that FA is wide from pure, which is $21.5 - x = 5.9$ where x is the sum of the narrowings of the P5s spanning FA. So $x = 21.5 - 5.9 = 15.6$. There are 4 P5/P4s spanning FA so each will be narrow by $15.6 / 4$ cent which is 3.9, i.e., 1/6' instead of 1/7'. This means the 4 P5s have eaten up 4/6' and as AE is

tuned the same this will be another $1/6'$. What remains is $1/6'$ which we distribute over EBF#C#G#, so each of those will now be $1/24'$ narrow.

The resulting circle of fifths diagram is depicted in Fig. 15 and it looks nice enough that I have computed the offsets and tuned it with good results.

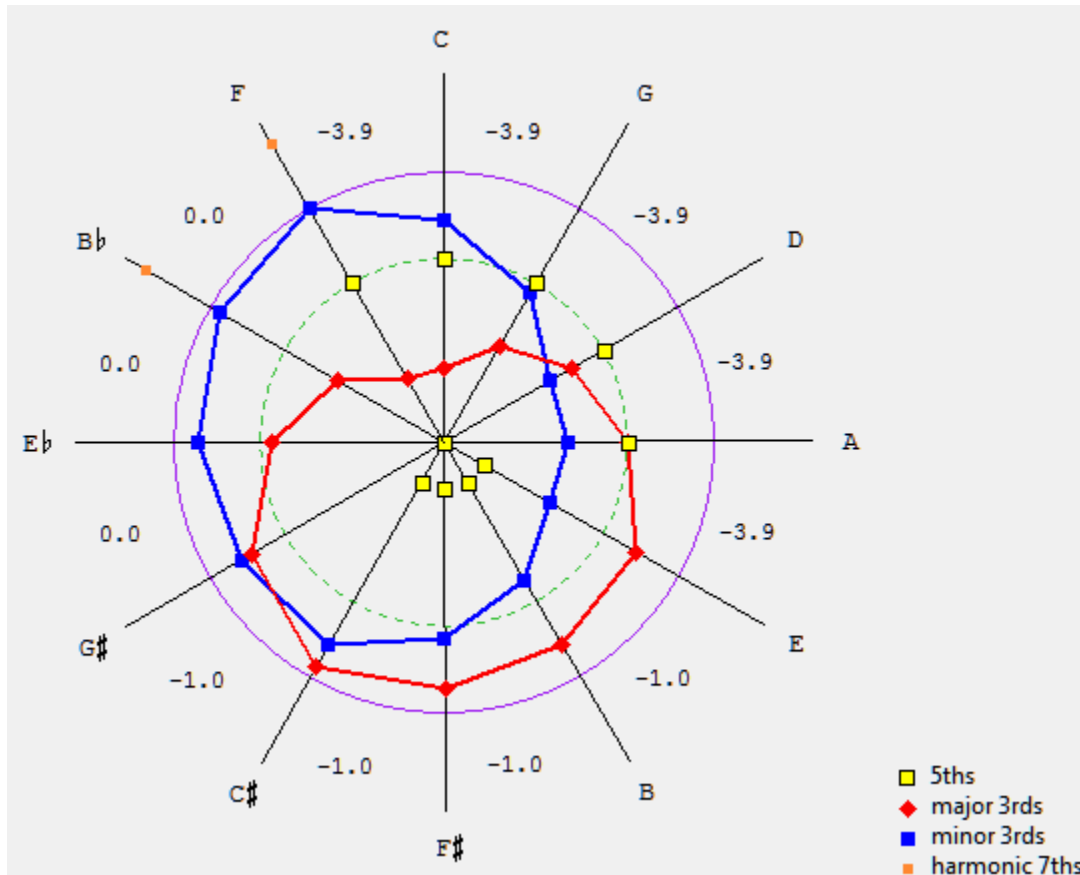


Figure 15. Reverse Bach-Lehman 14 tweaked to have FA beating at 3 bps instead of 4 bps.

Next suppose you like Reverse Lehman-Bach 14, but would like FA to be a bit less pure and you decide to tune it at 5 bps instead of at 4 bps and then follow the tuning recipe? That's also perfectly fine and results in another fine temperament that you can just tune and play. However let's analyse it, which is easy as I just copy the previous verbiage and change some numbers. We can use eq. 1, plug in bps (=5) and calculate cent offset of FA as $c = 1731 \cdot \text{bps} / f$ which gives $1731 \cdot 5 / 880 = 9.8$. This is the cents that FA is wide from pure, which is $21.5 - x = 9.8$ where x is the sum of the narrowings of the P5s spanning FA. So $x = 21.5 - 9.8 = 11.7$. There are 4 P5/P4s spanning FA so each will be narrow by $11.7 / 4$ cent which is 2.9, i.e., about $1/8'$ instead of $1/7'$. This means the 4 P5s have eaten up $4/8'$ and as AE is tuned the same this will be another $1/8'$. What remains is $3/8'$ which we distribute over EBF#C#G#, so each of those will now be $3/32'$ narrow. That's aurally indistinguishable from pure.

The resulting circle of fifths diagram is depicted in Fig. 16 and it also looks nice enough that I will compute the offsets and tune it.

In ET FA beats at 7 bps, so what would happen if you just tuned FA that way and follow the recipe? This is not a good idea, but let's go through the exercise just for fun.

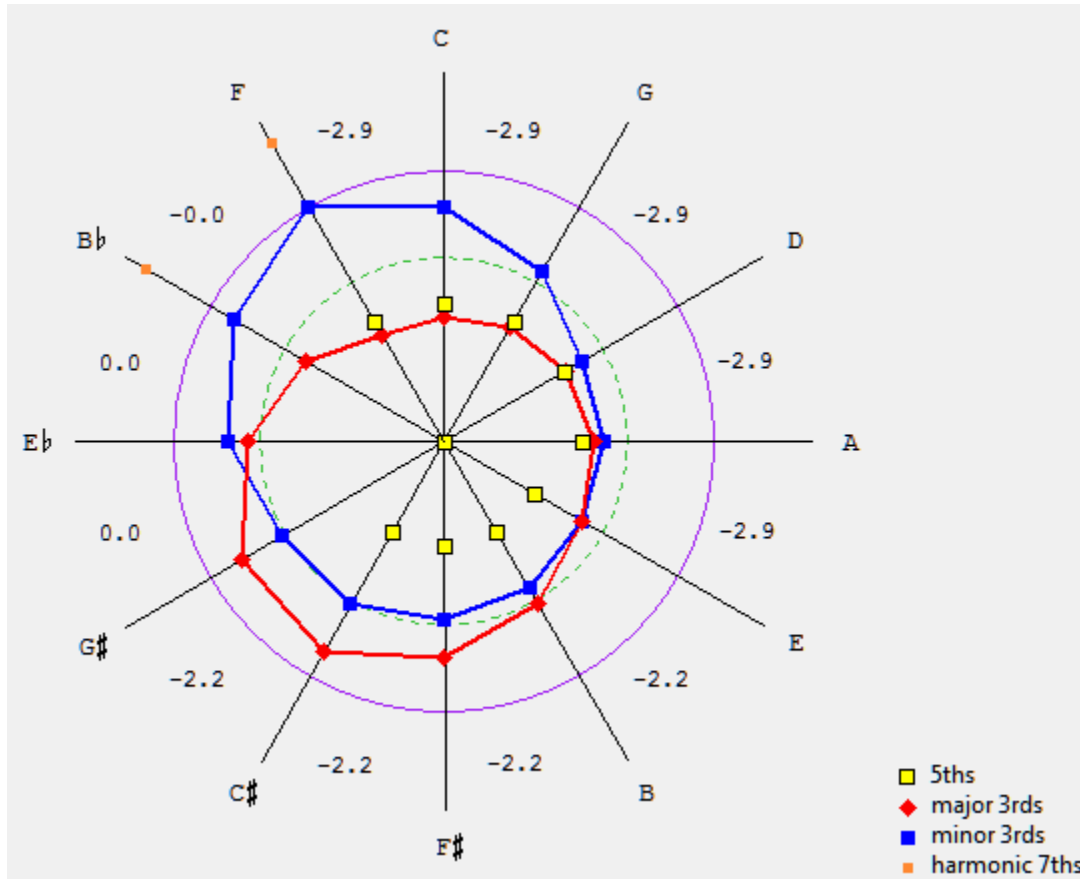


Figure 16. Reverse Bach-Lehman 14 tweaked to have FA beating at 5 bps instead of 4 bps.

We modify Reverse Lehman-Bach 14, but tune FA to at 7 bps instead of at 4 bps and then follow the tuning recipe. That's a bad idea, but let's show why as an exercise in analysis. This is easy as I just copy the previous verbiage and change some numbers. We can use eq. 1, plug in bps (=7) and calculate cent offset of FA as $c = 1731 * \text{bps} / f$ which gives $1731 * 7 / 880 = 13.8$. This is the cents that FA is wide from pure, which is $21.5 - x = 13.8$ where x is the sum of the narrowings of the P5s spanning FA. So $x = 21.5 - 13.8 = 7.7$. There are 4 P5/P4's spanning FA so each will be narrow by $7.7 / 4$ cent which is 2, i.e., about $1/12'$ instead of $1/7'$. This means the 4 P5s have eaten up $4/12'$ and as AE is tuned the same this will be another $1/12'$. What remains is $7/12'$ which we distribute over EBF#C#G#, so each of those will now be $7/48'$ narrow. That's aurally quite distinguishable from pure.

The resulting circle of fifths diagram is depicted in Fig. 17 and it looks ugly enough that I will not compute the offsets, let alone tune it.

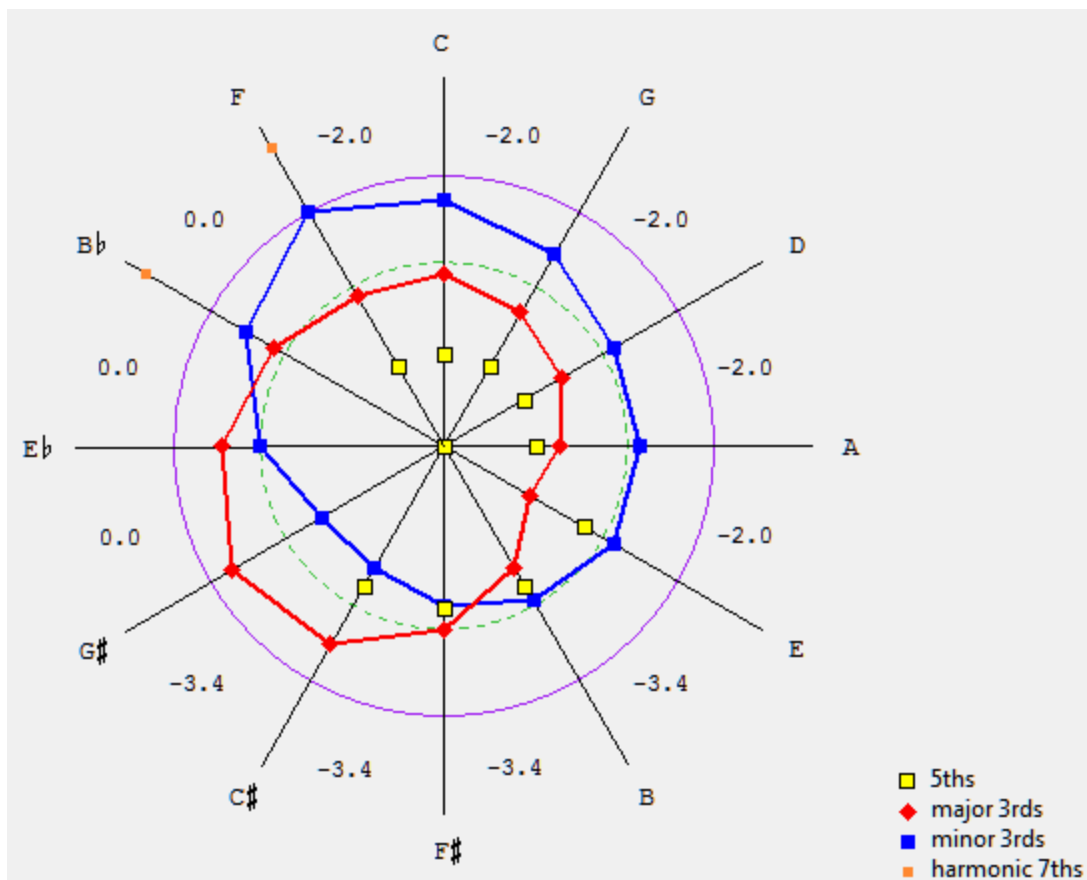


Figure 17. Reverse Bach-Lehman 14 tweaked to have FA beating at 7 bps instead of 4 bps.

At this point you may wonder, what about FA = 6 bps. Let's figure it out.

We modify Reverse Lehman-Bach 14, but tune FA to at 6 bps instead of at 4 bps and then follow the tuning recipe. That's an unclear idea, but let's figure out if it's reasonable in theory before tuning it. This is easy as I just copy the previous verbiage and change some numbers. We can use eq. 1, plug in bps (=6) and calculate cent offset of FA as $c = 1731 \cdot \text{bps} / f$ which gives $1731 \cdot 7 / 880 = 11.8$. This is the cents that FA is wide from pure, which is $21.5 - x = 11.8$ where x is the sum of the narrowings of the P5s spanning FA. So $x = 21.5 - 11.8 = 9.7$. There are 4 P5/P4s spanning FA so each will be narrow by $9.7 / 4$ cent which is 2.5, i.e., about $5/48'$ instead of $1/7'$. This means the 4 P5s have eaten up $5/12'$ ($=20/48'$) and as AE is tuned the same this will be another $5/48'$. What remains is $23/48'$ which we distribute over EBF#C#G#, so each of those will now be $23/192'$ narrow. That's aurally indistinguishable from pure.

The resulting circle of fifths diagram is depicted in Fig. 18 and it does not look terrible, but I am not tempted to tune this way, though I might just to try it out.

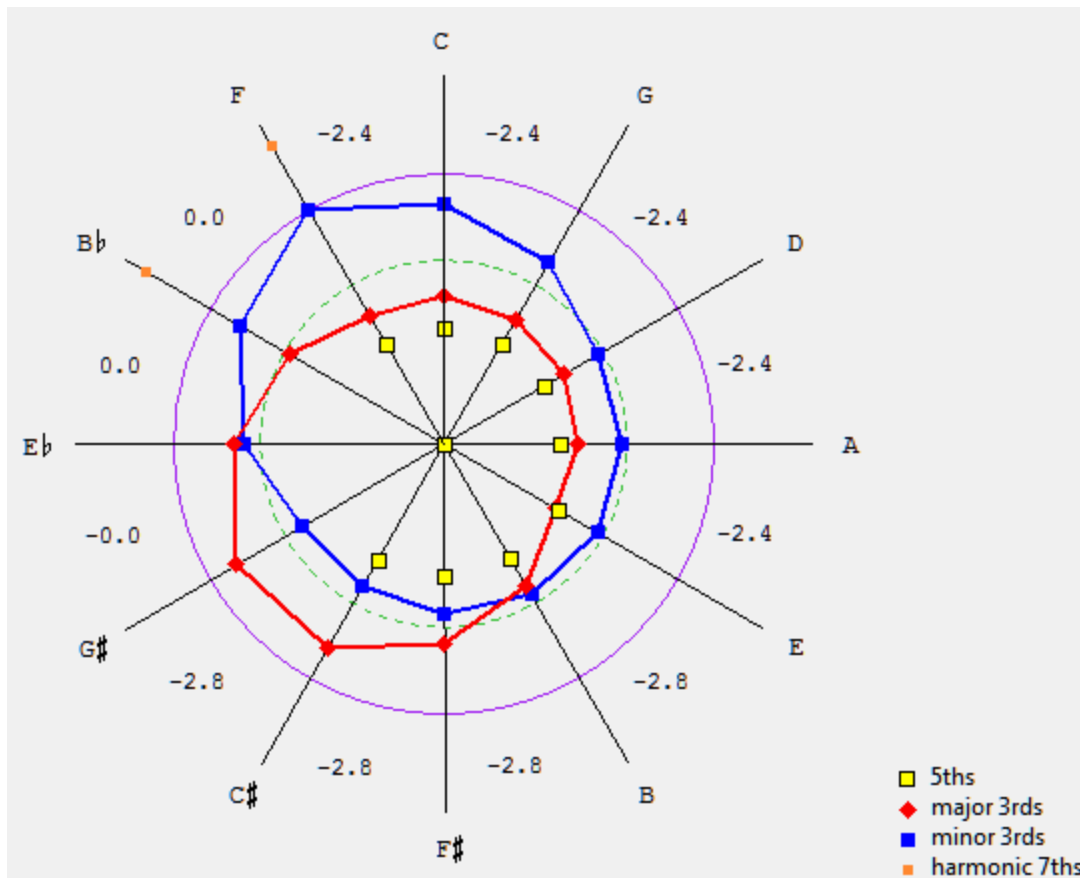


Figure 18. Reverse Bach-Lehman 14 tweaked to have FA beating at 6 bps instead of 4 bps.

Constructing circle of fifths from offsets

Frequently we encounter temperaments specified by offsets from ET. Here I will show how to calculate the sizes of the P5s from the offsets and make a circle of fifths diagram to analyse the temperament. We shall use a real-life example as depicted in Fig. 19. Those offsets were found on an organ in the San Francisco Bay area.

Let's start at F, though we could start anywhere. F is 4 cent sharp and C 6 cent, so the P5 FC is 2 cents wider than in ET, where it is 2 cent narrow, so it is a pure P5. G is +2 cent, C +6 cent so CG is 4 cent narrower than in ET (2 cent) so it is narrow by 6 cent (or $\frac{1}{4}$ ' as 2 cent is $\frac{1}{12}$ '). D is +2, same as G, so GD is as in ET, 2 cent narrow. D is +2, A is 0, so 2 cents narrower than ET, i.e., 4 cent (or $\frac{1}{6}$ '). And so on. The resulting temperament is depicted in Fig. 20. It looks quite bizarre.

Most likely the offset for G should be +4 instead of 2; in that case the famous Neidhardt 3 temperament is produced, depicted in Fig. 21. When the actual pitches of the organ were measured the G was indeed corrected to +4 instead of +2, but F was 2 instead of +4 and F# was +4 instead of +2. That temperament is depicted in Fig. 22. It is even more bizarre with a wide P5 on FC.

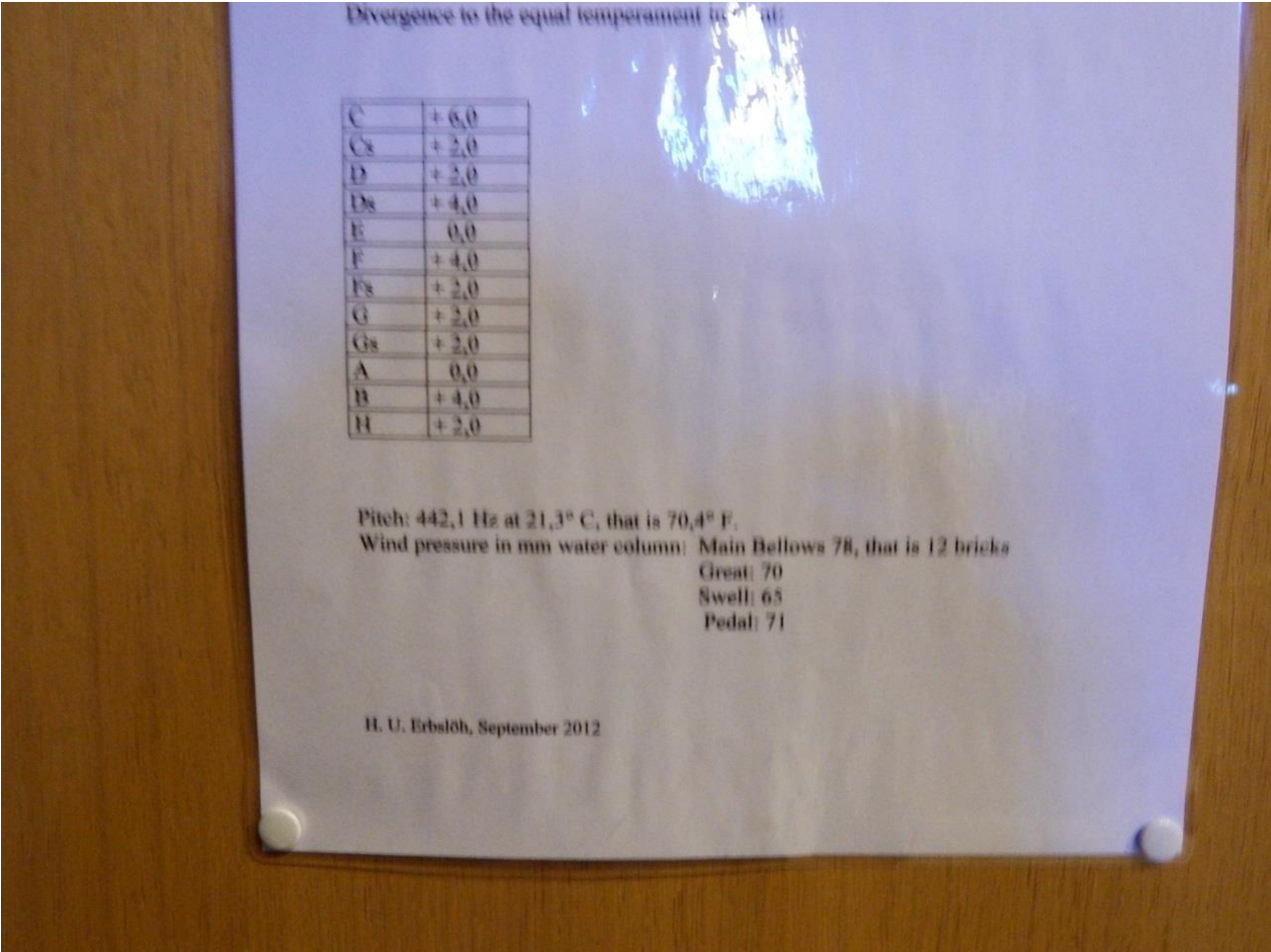


Figure 19. Offsets from ET of some organ in some temperament.

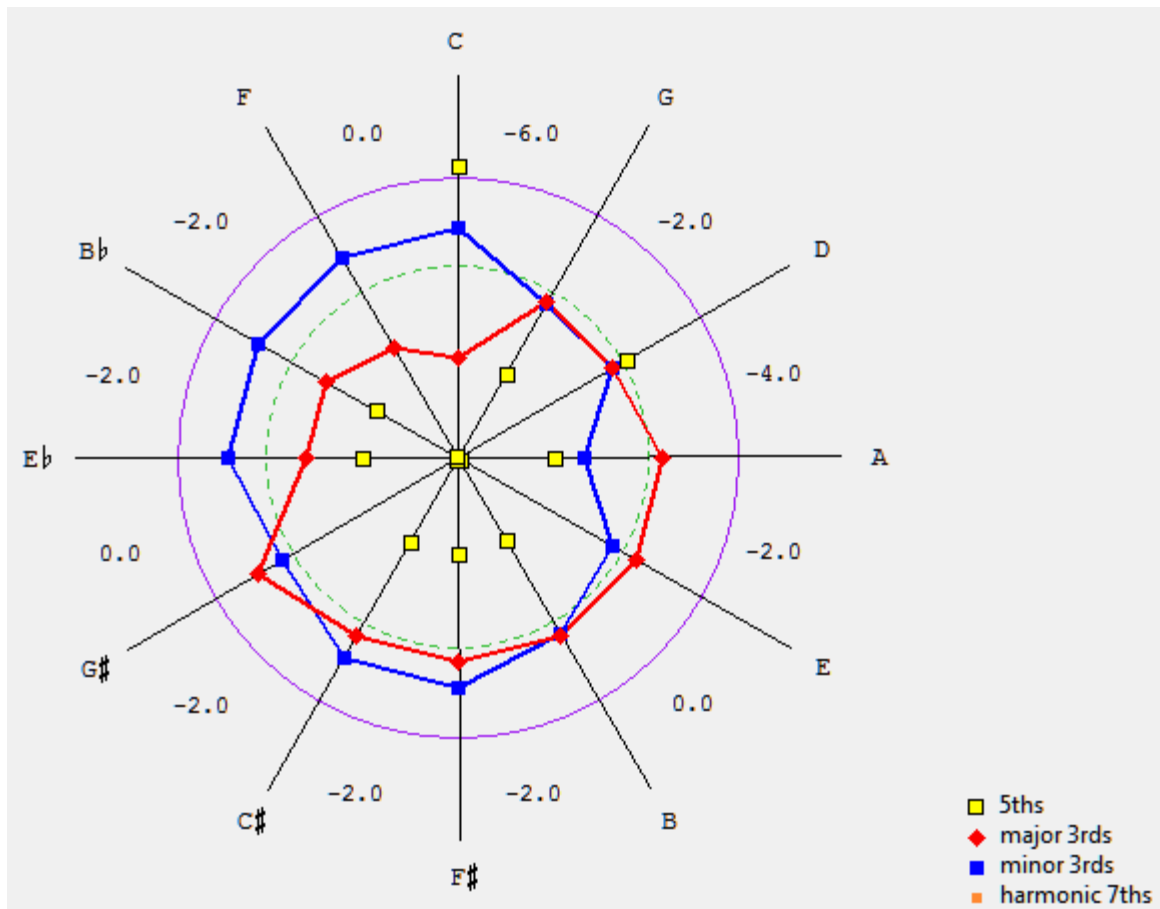


Figure 20. Temperament resulting from offsets in Fig. 19.

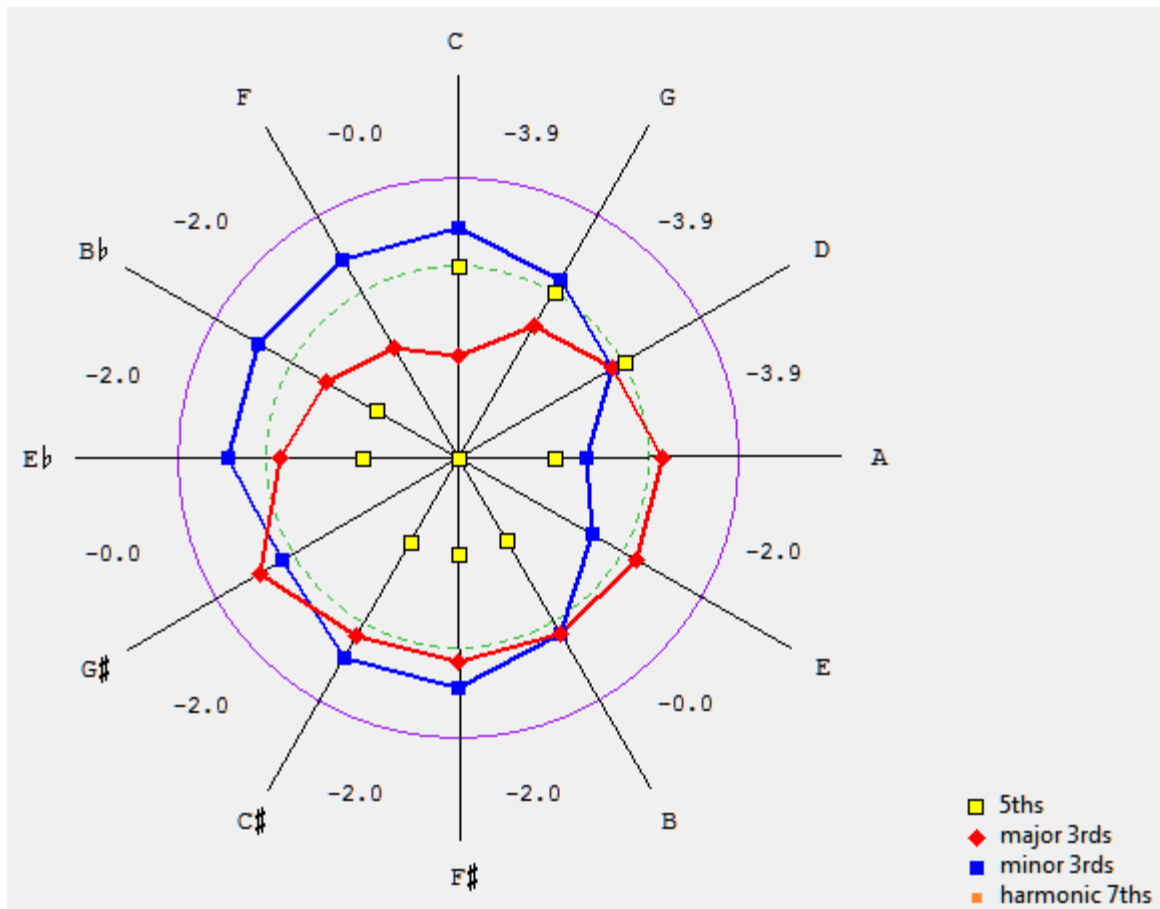


Figure 21. Neidhardt 3 (Grosse Stadt).

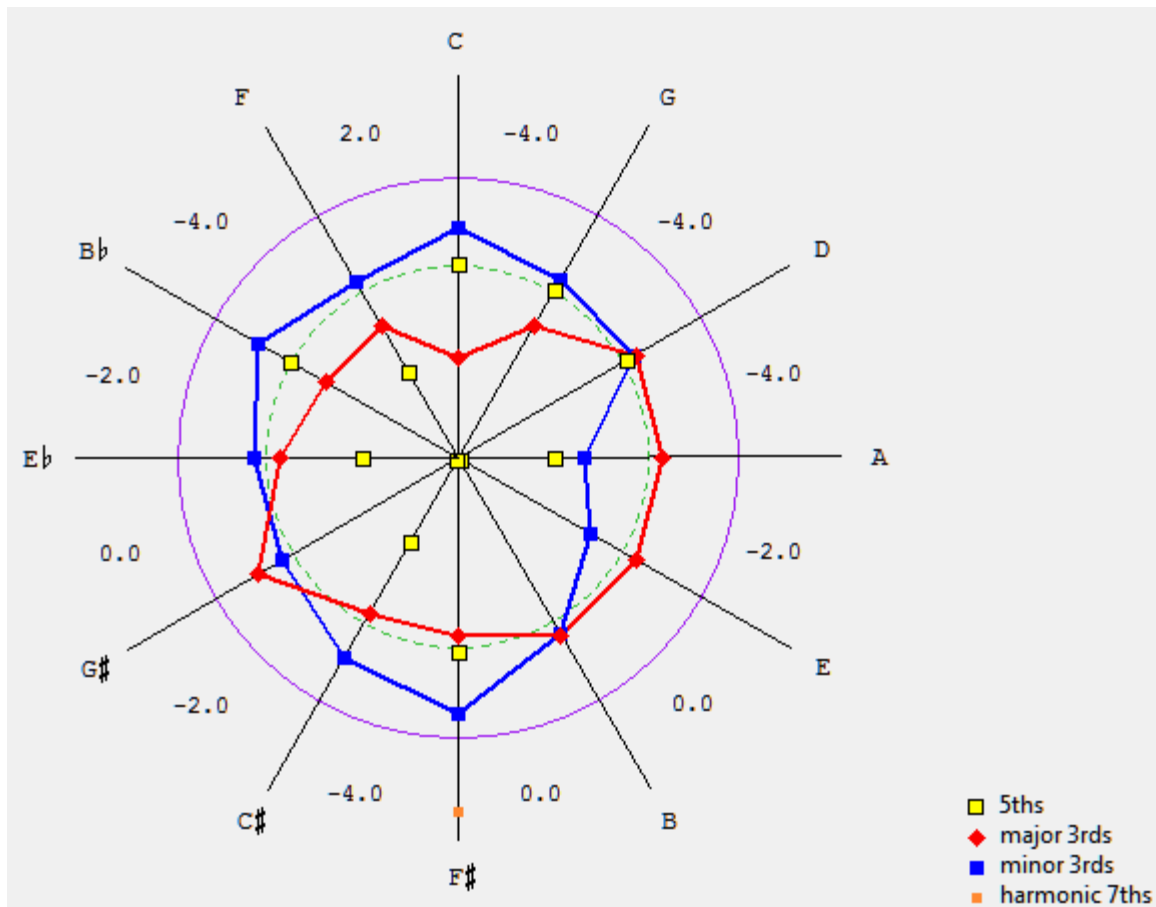


Figure 22. Actual measured temperament.

Designing a “perfect” temperament

The temperaments we have discussed thus far have at most two different sizes of P5s, apart from pure ones. This facilitates aural tuning as we can “fill in” the P5 and P4s from a given M3 beat speed by approximate equal beating intervals. However, for tuning with an electronic tuning device (ETD), any temperament is equally easy to tune when the offsets from ET are given. So perhaps it is possible to improve on historical temperaments by allowing a variety of P5 sizes. The following will be somewhat mathematical.

A temperament is defined by 12 P5 size, but since they have to add up to a negative Pythagorean comma, we have only 11 variables to choose, or “knobs” to turn. Naively one could then suppose we could set 11 M3 desired sizes and figure out what P5 sizes are necessary to achieve that. For example if we’d choose all M3s to be equal we should get ET. This is however not the case. A counterexample is depicted in Fig. 23 which has ET M3s but unequal P5s. Musically this is evident by considering some tuning and raising for example C, E, and G# by 20 cent: all M3s will stay the same. With some algebra it can be shown that the M3s only have 8 degrees of freedom. To get around these complications I will formulate the optimal temperament as a function optimization problem. This means that I’m going to

construct a mathematical function $f(p)$, where p stands for the 12 P5 sizes, in such a way that if we could make $f(p)=0$ all our desires regarding the temperament would be fulfilled.

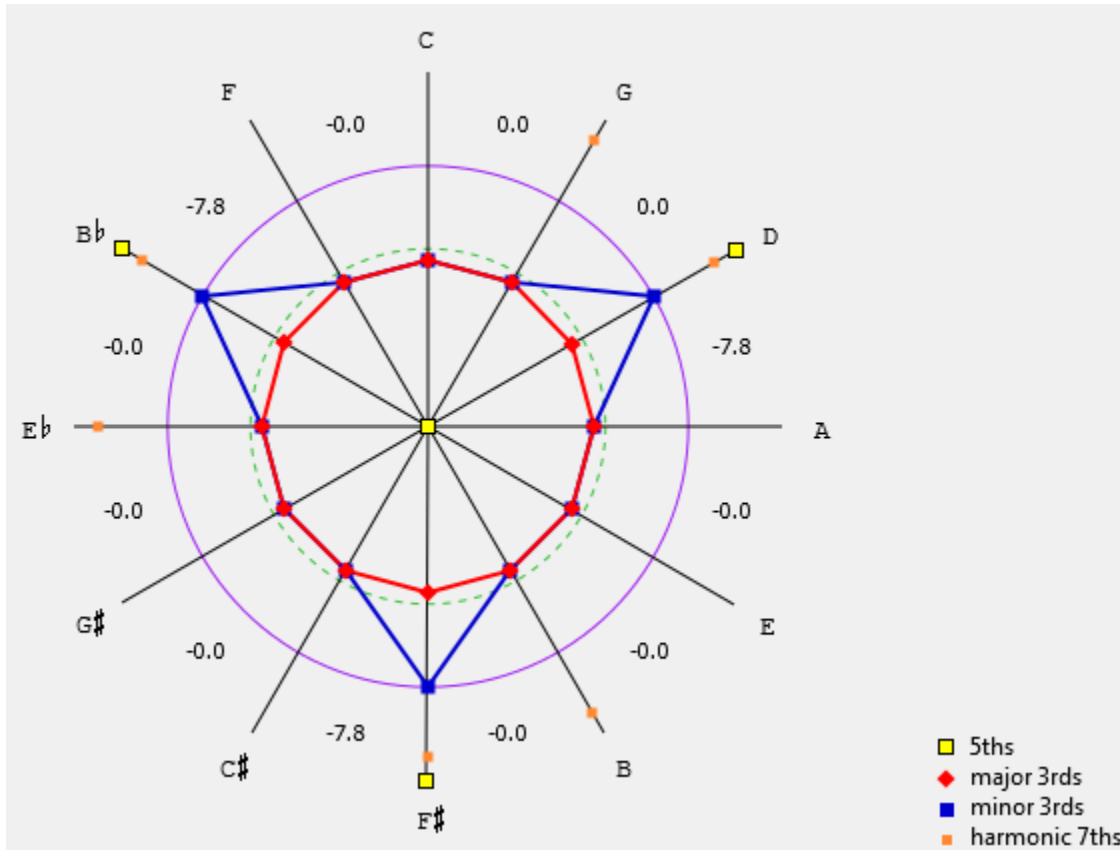


Figure 23. Marpurg temperament.

We will put 3 hard constraints on p , namely that they sum to a negative PC, that the M3 sizes of F and C are the same, and that the beat rate of F3A3 is set to a desired number like 4 bps for example. Now all we have to do is find the p that minimizes $f(p)$ subject to the three constraints just mentioned. That is, once you have defined $f(p)$, which I'll do now. We'd like C# to have the worst M3 (i.e., favouring sharps a bit) and have the M3 sizes from C to C# clockwise around the circle of P5s increase smoothly. Furthermore we'd like a similar smooth increase going counter clockwise from F to C#. We do this by trying to get the G M3 to be the mean of the M3s of F and D, the A M3 to be the mean of the M3s of D and B, the E M3 to be the mean of the M3s of A and B, the B M3 to be the mean of the M3s of E and F#, and the F# M3 to be the mean of the M3s of B and C#. Similarly we want the Bb M3 to be the mean of the M3s of F and Eb, the Eb M3 to be the mean of the M3s of Bb and G#, and the G# M3 to be the mean of the M3s of Eb and C#. The function $f(p)$ is now defined by computing how far off they are from the desired mean, squaring the number, and adding all 9 terms for GDAEBF#BbEbG#. One small problem remains: there may be multiple patterns of P5s that give the same result. To deal with this we add a small term to $f(p)$ which is the sum of the squares of the sizes of the P5s, divided by 10000. This steers the solution to the one with the purest P5s.

The constrained optimization problem was solved in MATLAB. In Fig. 24 we depict the resulting temperament setting F3A3 at 4 bps.

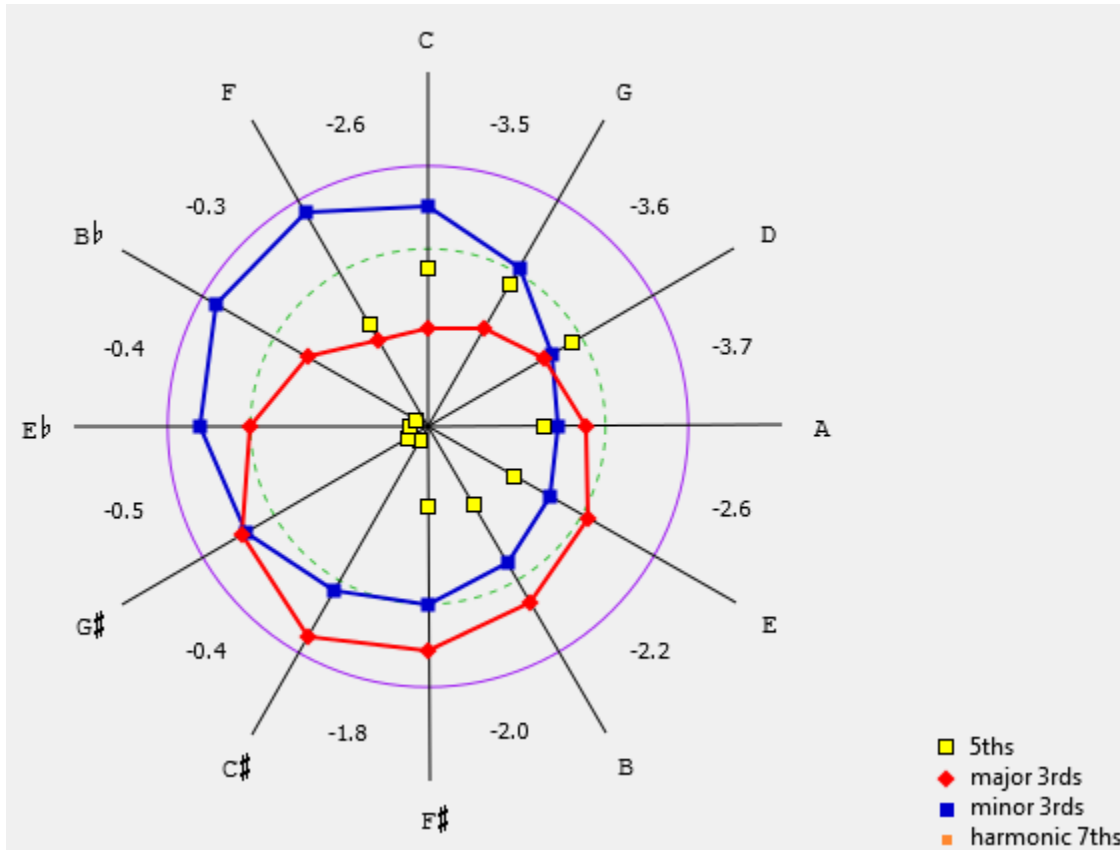


Figure 24. Optimal temperament with F3A3 = 4 bps.

It looks very similar to reverse Lehman-Bach 14, to the point of being indistinguishable. In Fig. 25 I show both diagrams combined. The ET offsets differ by less than a cent, and it is doubtful that there is an audible difference, even if a tuner could tune accurately enough to separate them. Also it turns out if we set F3A3 = 3 bps we pretty much recover the temperament from Fig. 15.

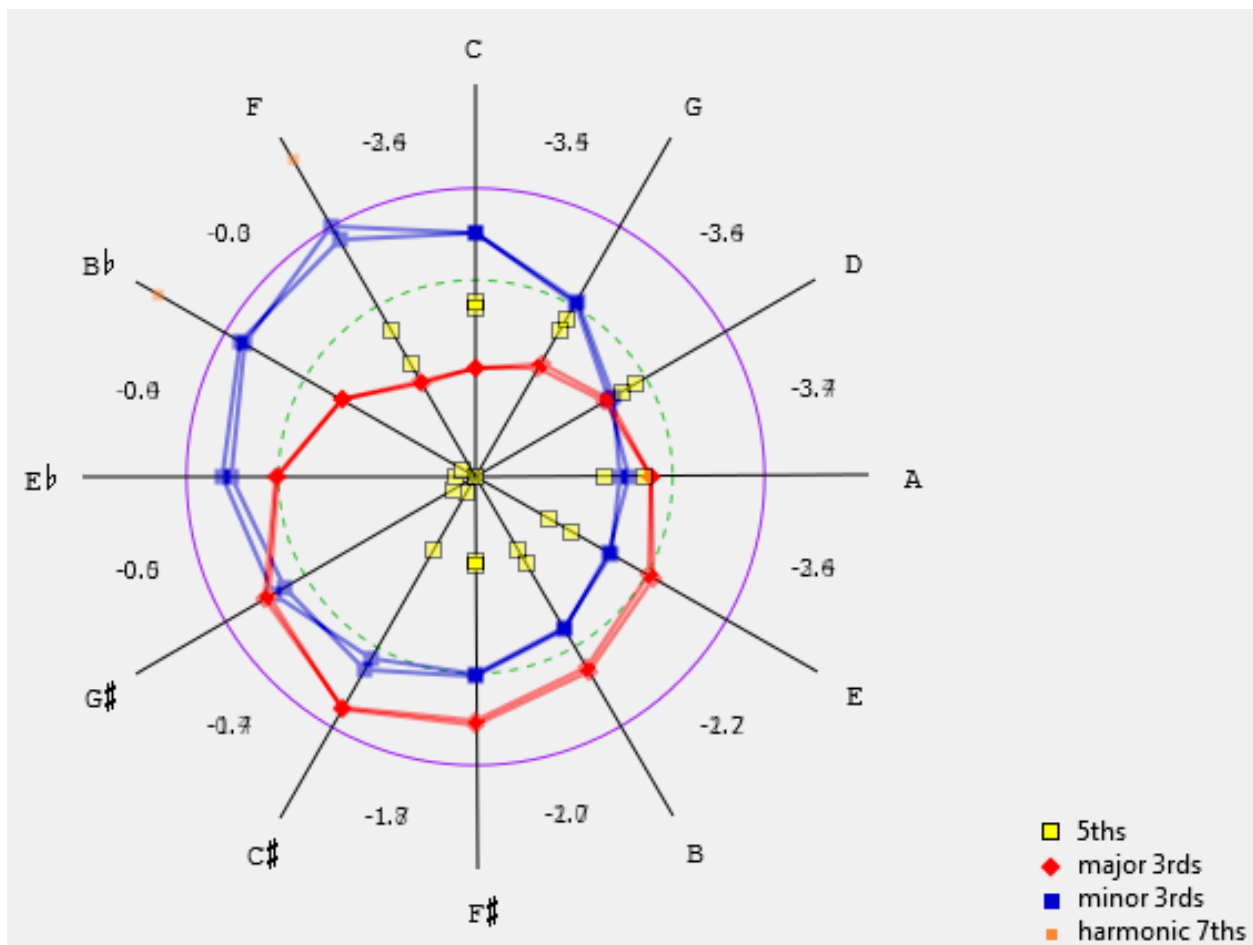


Figure 25. Optimal temperament with F3A3 = 4 bps mixed with reverse Lehman-Bach 14.