Octaves and inharmonicity in piano tuning

Kees van den Doel

January 11, 2013

Abstract

I show that equal beating 4:2/6:3 octaves (called 4:2+ octaves) results in an approximately equal beating 2:1 octave as well as long as the inharmonicity is increasing when going up, i.e., above the break. The consequences for 4:1 and 8:1 double and triple octave beat rates is also analysed.

1 Partials, beats, and cents

Let's start by reviewing some basic formulae and definitions.

Suppose we have two single frequencies (sine waves) x and y, measured in Hertz (or any other units). The cent distance c between y and x is per definition

$$c = \frac{1200}{\log 2} \log(y/x), \tag{1}$$

with log the natural logarithm. If x and y are very close we will hear a beat with frequency (beatrate) $b = |y - x|$. The beatrate is given very accurately in terms of the cent difference by

$$b = \frac{\log 2}{1200} cx. \tag{2}$$

For example is $x = 440Hz$ and $c = 1$ we get $b \approx 1/4$, i.e., one beat every 4 seconds for a 1 cent detuned $A4$ unison.

A real piano tone is composed of not just a single frequency but a set of partials. In the absense of inharmonicity the partial frequencies are $x, 2x, 3x, \ldots$ with x the fundamental frequency. In terms of cents the distance of the i-th partial from the fundamental is given by

$$c(i) = \frac{1200}{\log 2} \log(i), \quad i = 1, 2, 3, \ldots \tag{3}$$

In a real piano the partials are all shifted up by a small amount, and effect called inharmonicity. We consider here the Young model [2] and a more realistic modification thereof based on piano data by Robert Scott as implemented in Tunelab [1]. According to this model each partial i is offset by an amount $B(a_i - 1)$ where B is the inharmonicity constant of the note and the coefficients a_i are given by

$$a_i = i^2 \tag{4}$$

in the Young model and are given by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 8.45 \\ a_4 &= 13.18 \\ a_5 &= 19.72 \\ a_6 &= 27.27 \\ a_7 &= 35.53 \\ a_8 &= 46.25 \end{aligned} \tag{5}$$

by the Tunelab model. Values for higher partials can be found in the Tunelab manual. Formula (3) is thus modified to

$$c(i) = \frac{1200}{\log 2} \log(i) + B(a_i - 1). \tag{6}$$

2 Tuning the octave by partial matching

Consider tuning a note N_2 an octave above a note N_1. Both notes have a partial structure given by (6), but will have different inharmonicity constants B which we call B_1 and B_2. Above the break B_2 will be about twice the value of B_1.

Let's measure all the partials in cents relative to the fundamental of N_1. Also, let's assume we first tune N_2 so that its fundamental is exactly 1200 cents above the fundamental of N_1 and then adding a stretch of s_2 cents, where the value of s_2 is to be determined later. The cent values of the i-th partials of both notes (w.r.t the fundamental of N_1) are now given by

$$\begin{aligned} c_1(i) &= \frac{1200}{\log 2} \log(i) + B_1(a_i - 1) \\ c_2(i) &= 1200 + \frac{1200}{\log 2} \log(i) + B_2(a_i - 1) + s_2. \end{aligned} \tag{7}$$

As a warm-up excercise, let's compute the stretch s_2 first for a 2:1 octave. We want the second partial of N_1 to match the first partial of N_2, so the equation to solve for

s_2 is $c_1(2) = c_2(1)$ which reads

$$1200 + B_1(a_2 - 1) = 1200 + s_2, \tag{8}$$

where we have used $a_1 = 1$. We thus get for a 2:1 octave the stretch

$$s_2[2:1] = B_1(a_2 - 1). \tag{9}$$

For example if $N_1 = C4$ and using a typical value $B_1 = 0.404$ we get a stretch of 1.21 cents. We can calculate the theoretical stretch for the 4:2 and 6:3 octaves in the same manner by solving $c_1(4) = c_2(2)$ and $c_1(6) = c_2(3)$ which gives

$$s_2[4:2] = B_1(a_4 - 1) - 3B_2 \tag{10}$$

and

$$s_2[6:3] = B_1(a_6 - 1) - B_2(a_3 - 1). \tag{11}$$

We see that the larger B_2 is, i.e., the steeper the inharmonicity curve is, the less stretch is required for these octaves. Taking the "average" values for C4 and C5 from the Tunelab sample data, $B_1 = 0.404$ and $B_2 = 1.116$ this gives stretches of $s_2[4:2] = 1.57$ and $s_2[6:3] = 2.3$ cents.

3 Tuning the octave by equal beating 4:2 and 6:3

Next we will compute the stretch required for the 4:2+ octave, which has equal beating 4:2 and 6:3, with 4:2 being wide, and 6:3 narrow. First we need the cent difference $\Delta(i)$ between partial i of N_2 and partial $2i$ of note N_1. Using (7) we get

$$\Delta(i) = c_2(i) - c_1(2i) = B_2(a_i - 1) - B_1(a_{2i} - 1) + s_2. \tag{12}$$

A positive $\Delta(i)$ means the $2i : i$ octave is wide, a negative means it is narrow. The beat rate $b(i)$ of this difference according to (2) is given by

$$b(i) = \frac{\log 2}{1200}\Delta(i)2if_0 \tag{13}$$

where f_0 is the frequency of the fundamental of N_1. Note that the frequency of partial $2i$ is not exactly $2if_0$ but the difference is very small and lies beyond the threshold of hum beat speed discrimination. The beat frequency (or rate) of the $2i : i$ octave as given by (13) is positive if the octave is wide, and negative if narrow. We now want to find s_2 such that $b(2) = -b(3)$, i.e., the 4:2 beats wide at the same speed the 6:3 beat narrow. Using (13) and (12) we obtain the stretch s_2

$$\begin{aligned}
&b(2) + b(3) = 0 \;\Rightarrow \\
&2\Delta(2) + 3\Delta(3) = 0 \;\Rightarrow \\
&2B_2(a_2 - 1) - 2B_1(a_4 - 1) + 2s_2 + 3B_2(a_3 - 1) - 3B_1(a_6 - 1) + 3s_2 = 0 \;\Rightarrow \\
&s_2[4:2+] = (B_1(2a_4 + 3a_6 - 5) - B_2(2a_2 + 3a_3 - 5))/5.
\end{aligned} \tag{14}$$

If we use the numbers as after (11) we obtain a stretch of 2 cents. However if we used Young's model the stretch would come out to be about 4.2 cents, which is more than twice as much!

With the 4:2+ stretch (14) in had we can now compute the beat rates of the various octave partial matches by using (12) and (13). After some algebra we get the 4:2+ octave 2i:i beat rates

$$b(i) = \frac{f_0 \log 2}{3000}[B_1(2a_4 + 3a_6 - 5a_{2i}) - B_2(2a_2 + 3a_3 - 5a_i)]i. \tag{15}$$

For the Tunelab model this comes out to be for 4:2 as

$$b(2) = b_{4:2} = \frac{f_0 \log 2}{1200}(33.82B_1 - 10.7B_2). \tag{16}$$

Of course be have $b(2) = -b(3)$ which is easy to check. The big question is now, what about 2:1? So let's compute the beat rate difference between 4:2 and 2:1, i.e., $b(2) - b(1)$ using (15). We get

$$b(2) - b(1) = \frac{f_0 \log 2}{3000}[B_1(2a_4 + 3a_6 + 20 - 10a_4) - B_2(2a_2 + 3a_3 - 35)]. \tag{17}$$

Remarkably, for Young's model $a_i = i^2$ the coefficients of B_1 and B_2 are both zero so we have 2:1 beating at exactly the same speed as 4:2 and 6:3. For the tunelab model, plugging in in the table values we obtain instead

$$b(2) - b(1) = \frac{f_0 \log 2}{3000}[1.65B_2 - 3.63B_1]. \tag{18}$$

So if $B_2 = 2.2B_1$ we have 2:1 equal beating, and this relation is almost satisfied on most piano scales. Even if it were not and we had $B_2 = B_1$ or $B_2 = 4B_1$ which are extreme values probably not found on any real piano, we still have a beat speed difference of only 0.1 and 0.14 Hz (or beats per second if you like): virtually equal beating! For comparison the 4:2 (and 6:3) beat speed in this example is 0.5 Hz. On the other hand, the 8:4 octave is quite narrow and beats at 6.4 Hz.

4 Double and triple octaves

Let us now imagine tuning N_2 to N_1 as a 4:2+ octave, and next N_3 to N_2 and N_4 to N_3 with the same method. What will the double octaves (4:1) and triple octave (8:1) be like?

To analyze this we extend (7) for the notes $N_1, \dots, N_4$, measuring all frequencies in cents relative to the fundamental of N_1 for analysis purposes. As the octave stretches are cumulative we now obtain the following expression for the i-th partial of octave k

$$c_k(i) = 1200(k-1) + \frac{1200}{\log 2}\log(i) + B_k(a_i - 1) + \sum_{j=1}^{k} s_j. \tag{19}$$

where s_j is the stretch of octave $N_j - N_{j-1}$ for $j > 1$ and for notational convenience we define $s_1 = 0$. The individual octave stretches are obtained from (14) as

$$s_k = [B_{k-1}(2a_4 + 3a_6 - 5) - B_k(2a_2 + 3a_3 - 5)]/5, \quad k > 1, \tag{20}$$

where B_k is the inharmonicity constant of note N_k. To simplify notation let's write (20) as

$$s_k = q_1 B_{k-1} - q_2 B_k, \quad k > 1, \tag{21}$$

with

$$\begin{aligned} q_1 &= (2a_4 + 3a_6 - 5)/5 = 20.634\ (27) \\ q_2 &= (2a_2 + 3a_3 - 5)/5 = 5.67\ (6), \end{aligned} \tag{22}$$

where the numerical values are according to the Tunelab model and the parenthesized values according to Young's model.

It is now easy to obtain the 4:1 beat rate (of N_1 versus N_3) as

$$b_{4:1} = \frac{\log 2}{300} f_0[c_1(4) - c_3(1)] = \frac{\log 2}{300} f_0[s_2 + s_3 - B_1(a_4 - 1)]. \tag{23}$$

Substituing the values from (21) gives

$$b_{4:1} = \frac{\log 2}{300} f_0[c_1(4) - c_3(1)] = \frac{\log 2}{300} f_0[(q_1 + 1 - a_4)B_1 + (q_1 - q_2)B_2 - q_2 B_3]. \tag{24}$$

Numerically using the Tunelab model this gives

$$b_{4:1} = \frac{f_0 \log 2}{1200}(33.82B_1 + 59.9B_2 - 22.7B_3) \tag{25}$$

to be compared with for example the 4:2 beat rate of the N_2N_3 single octave given by (16) which gives

$$b_{4:2} = \frac{f_0 \log 2}{1200} 2(33.82B_2 - 10.7B_3). \tag{26}$$

(Note the extra factor 2 as we are considering the second single octave.)

Let's take $(B_1, B_2, B_3) = (0.404, 1.116, 2.12)$ from the Tunelab "average" tuning file, corresponding to C4-C5-C6. Unfortunately both C5-C6 4:2 and the C4-C6 4:1 beat rates are about 9 Hz (both wide) which is much too high to be acceptable. The C4-C5 4:2 beat rate remains good of course, about 0.5 Hz. The reason is that the steeper the slow of the inharmonicity curve, the better the 4:2 octave is as follows from (26), up to a limit of $B3 \approx 3B2$ where both 4:2 and 6:3 become pure (at the expense of 2:1).

As a second example let us consider the octaves F3F4F5F6 tuned according to 4:2+ for the case of a Kawai K3 for which I have the inharmonicity constants which are $(0.372, 0.566.1.56, 3.6)$. Plugging in the numbers we get an F3F4 4:2 beat rate of 0.5 Hz, an F4F5 4:2 beat rate of 5.7 Hz, and 33Hz for F5F6. The 4:1 beat rates are 1 Hz for F3F5 and 6 Hz for F4F6.

5 Conclusions

The equal beating 4:2/6:3 octave (4:2+) leads to virtual identical beating rates of also the 2:1 octave. Whenever the beat rate is slow in the absolute sense the theory therefore support the notion that this is the aurally purest octave size.

However, this seems to apply only to the temperament octave, as outside the temperament octave as it leads to much too fast beating 4:2 octaves and 4:1 double octaves. Above this area the stretch has to be reduced, sacrificing the 6:3 octave (which rapidly becomes too high pitched to be relevant anyways) for a slower 4:2 beat.

References

[1] Robert Scott. Tunelab piano tuning software. *http://www.tunelab-world.com*.

[2] Robert W. Young. Inharmonicity of Plain Wire Piano Strings. *The Journal of the Acoustical Society of America*, 24(3), 1952.